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Weak Continuity of the Cartan Structural System and Compensated Compactness on Semi-Riemannian Manifolds with Lower Regularity

Gui-Qiang G. Chen · Siran Li

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Abstract We are concerned with the global weak continuity of the Cartan structural system — or equivalently, the Gauss–Codazzi–Ricci system — on semi-Riemannian manifolds with lower regularity. For this purpose, we first formulate and prove a geometric compensated compactness theorem on vector bundles over semi-Riemannian manifolds with lower regularity (Theorem 3.2), extending the classical quadratic theorem of compensated compactness. We then deduce the L^p weak continuity of the Cartan structural system for $p > 2$: For a family $\{\mathcal{W}_\varepsilon\}$ of connection 1-forms on a semi-Riemannian manifold (M, g), if $\{\mathcal{W}_\varepsilon\}$ is uniformly bounded in L^p and satisfies the Cartan structural system, then any weak L^p limit of $\{\mathcal{W}_\varepsilon\}$ is also a solution of the Cartan structural system. Moreover, it is proved that isometric immersions of semi-Riemannian manifolds into semi-Euclidean spaces can be constructed from the weak solutions of the Cartan structural system or the Gauss–Codazzi–Ricci system (Theorem 5.1), which leads to the L^p weak continuity of the Gauss–Codazzi–Ricci system on semi-Riemannian manifolds. As further applications, the weak continuity of Einstein's constraint equations, general immersed hypersurfaces, and the quasilinear wave equations is also established.

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G.-Q. G. Chen
Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK
E-mail: chengq@maths.ox.ac.uk

S. Li
New York University–Shanghai, Office 1146, 1555 Century Avenue, Pudong District, Shanghai 200122, China, and NYU–ECNU Institute of Mathematical Sciences, Room 340, Geography Building, 3663 North Zhongshan Road, Shanghai 200062, China
E-mail: sl4025@nyu.edu
Present address: School of Mathematical Sciences, Shanghai Jiao Tong University, No. 6 Science Buildings, 800 Dongchuan Road, Minhang District, Shanghai 200240, China

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1 Introduction

We are concerned with isometric immersions of semi-Riemannian manifolds with arbitrary signature into semi-Euclidean spaces. We establish the weak continuity of two fundamental systems of nonlinear partial differential equations (PDEs): the Cartan structural system and the Gauss–Codazzi–Ricci (GCR) system, which constitute the compatibility equations for the existence of isometric immersions.

The isometric immersion problem has been of fundamental importance in the development of modern differential geometry. It has led to various new techniques and ideas in nonlinear PDEs, nonlinear analysis, and geometric analysis (*cf.* [8,33,34,63] and the references cited therein). On the other hand, it has wide applications. For example, in theoretical physics, the manners in which our 4-dimensional space-time is immersed in the ambient universe correspond to different cosmological models (*cf.* Mars–Senovilla [45,46]), and the isometric immersion of round spheres into warped product manifolds is central to recent versions of quasi-local mass (*cf.* Guan–Lu [32] and Wang–Yau [62]). Moreover, the isometric immersions of semi-Riemannian manifolds *with lower regularity* are fundamental in many scientific areas. For example, such immersions arise in the thin-shell model for gravitational source and the junction condition for gluing disjoint space-times; see [2,19,29] for the details.

In the classical work [50], Nash established the existence of isometric embeddings of Riemannian manifolds with C^k metrics, $k \geq 3$, into the Euclidean spaces of high dimensions. The analogous problem for semi-Riemannian manifolds (*i.e.*, the metrics are not necessarily positive-definite) is posed as a natural extension. More importantly, the isometric immersion problem of semi-Riemannian manifolds is fundamental in general relativity and Lorentzian geometry. Clarke [18] proved the existence theorem of isometric embeddings of C^k semi-Riemannian manifolds into semi-Euclidean spaces, under additional hypotheses on the signature. Despite these general existence theorems, the analysis for isometric immersions of semi-Riemannian manifolds appears more challenging than its Riemannian analog. In particular, the Laplace–Beltrami operator is no longer elliptic, thus precludes the standard elliptic PDE machineries. See Goenner [30], Greene [31], and the references cited therein for the earlier rigorous mathematical analysis on isometric immersions of semi-Riemannian manifolds.

Motivated by both mathematical and physical importance discussed above, in this paper, we study the isometric immersions of semi-Riemannian manifolds with lower regularity. One of the fundamental tools for investigating the isometric immersions is the GCR system (*cf.* [8,13,14,30,36,41]), which describes the geometry of the ambient space in terms of the geometry of the tangential and normal directions of the immersed submanifold. We are interested in the global weak continuity of the GCR system, as well as the global weak rigidity of the corresponding isometric immersions and curvatures.

The analysis of the GCR system encompasses several challenges, primarily because they do not have a fixed type — elliptic, parabolic, or hyperbolic — in general. Even in the Riemannian case, when the immersed manifold has dimension higher than 3, it is proved by Bryant–Griffith–Yang [8] that the GCR system has no definite type. The novel observation by Chen–Slemrod–Wang in [13,14] (also see [12]) shows that the GCR system for Riemannian manifolds possesses an intrinsic *div-curl structure*, so that the compensated compactness techniques for nonlinear analysis can be applied, which is independent of the types of the system.

In order to employ the compensated compactness techniques in semi-Riemannian settings, however, we meet with further complications. First, the effective proofs of the div-curl lemma rely essentially on the ellipticity of the Laplace–Beltrami operator; *cf.* Evans [28], Robbin–Rogers–Temple [52], Kozono–Yanagisawa [39], Chen–Li [12], and the references cited therein. This does not hold for semi-Riemannian manifolds. Moreover, the non-trivial signatures of the semi-Riemannian metrics make it difficult to identify the div-curl structure globally.

To overcome the new complications, we further exploit the geometry of isometric immersions of semi-Riemannian submanifolds. Rather than tackling the GCR system directly, we first establish the weak continuity of the Cartan structural system. This is proved to be equivalent to the GCR system, even for the semi-Riemannian manifolds with lower regularity in $W^{2,p}$. The Cartan structural system possesses a natural quadratic structure. For this purpose, we first establish a global, intrinsic compensated compactness result (Theorem 3.2) in the setting of vector bundles over semi-Riemannian manifolds, and then apply it to give a rigorous proof of the weak continuity of the Cartan structural system. We emphasize the *global* and *intrinsic* nature of these results, in the sense that their formulations are independent of local coordinate systems.

The compensated compactness techniques have been developed in the study of nonlinear PDEs in the Euclidean space $\mathbb{R}^d$, especially for nonlinear conservation laws such as the Euler equations in fluid mechanics; see [11, 24,28] and the references therein. One of the major results in the theory of compensated compactness is the *quadratic theorem* in $\mathbb{R}^d$ (see Murat [49] and Tartar [59]). For our purpose, we establish a generalized quadratic theorem that is of global and intrinsic nature on vector bundles. Our crucial observation is that the first-order differential constraints in the quadratic theorem on $\mathbb{R}^d$ can be replaced by more general assumptions on the principal symbol of the associated differential operators, while the principal symbol is diffeomorphism-invariant on manifolds. This leads to an intrinsic formulation of the quadratic theorem on vector bundles over semi-Riemannian manifolds.

Other generalizations of the quadratic theorem were established in the literature. Mišur–Mitrović in [48] studied the weak convergence of quadratic expressions $\sum_{i,j=1}^{N} q_{ij} u_\varepsilon^i v_\varepsilon^j$, where $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ are weakly convergent in $L^p(\mathbb{R}^d;\mathbb{R}^N)$ and $L^{p'}(\mathbb{R}^d;\mathbb{R}^N)$, respectively, for $\frac{1}{p}+\frac{1}{p'}\leq 1$. For this, coefficients $q_{ij}, i,j=1,\cdots,N$, are allowed to depend on $x\in\mathbb{R}^d$, the conditions involve fractional derivatives, and the idea of H-distributions is used in the proof; also see §3 in Mišur [47]. In contrast, our generalized quadratic theorem is geometric and global in nature, which serves naturally for our purpose to establish the weak continuity of both the Cartan structural system and the GCR system.

The results and techniques established in this paper have applications to semi-Riemannian geometry, from the perspectives of both mathematics and physics. For example, we deduce the weak rigidity of isometric immersions of semi-Riemannian manifolds by using the weak continuity of the Cartan structural system or the GCR system. The realizability of isometric immersions of semi-Riemannian manifolds with lower regularity from the weak solutions of the Cartan structural system or the GCR system (Theorem 5.1) is proved along the way. In addition, we demonstrate the weak continuity properties of Einstein's constraint equations, quasilinear wave equations, and degenerate hypersurfaces in space-time.

We emphasize that, in this paper, we are concerned mainly with semi-Riemannian manifolds (M,g) with lower regularity, which means that M is parametrized by $W^{2,p}_{\rm loc}$ maps, or that metric g is in $W^{1,p}_{\rm loc}\cap L^\infty_{\rm loc}$. The weak continuity of the GCR system and the Cartan structural system is established in such regularity classes with $p>2$, regardless of the dimension of M. In particular, when $\dim M\geq 3$, these weak continuity results cannot be deduced from the realization theorem of isometric immersions from the GCR system, or equivalently the Cartan structural system, since it can be proved so far only under the *stronger assumption*: $p>\dim M$. This imposes considerable additional difficulties. In fact, apart from the realization theorem (Theorem 5.1), we will restrict ourselves only to $p>2$ (rather than $p>\dim M$) everywhere else throughout the paper.

We remark in passing that the $W^{2,p}$ continuity of the GCR and Cartan structural systems may also be established via computing carefully in local coordinate systems, by utilizing the compensated compactness techniques in the flat space $\mathbb{R}^d$ (see, *e.g.*, Chen–Slemrod–Wang [14] and Robbin–Rogers–Temple [52]). However, the compensated compactness results established in the semi-Riemannian setting in this paper not only provide a direct intrinsic proof of the $W^{2,p}$ continuity of the Cartan structural systems, but also are of independent interest. In particular, the global and intrinsic formulation of Theorem 3.2 contributes to the theory of compensated compactness and its further applications.

The rest of this paper is organized as follows: In §2, we review the Cartan structural system and the basics of the semi-Riemannian submanifold theory. The bundle-theoretic perspectives are emphasized. In §3, we establish a global intrinsic compensated compactness theorem on vector bundles over semi-Riemannian manifolds, which is also extended to locally compact Abelian groups. Employing the results in §3, we deduce the weak continuity of the Cartan structural system in §4. Next, in §5, we solve the realization problem (*i.e.*, the construction of isometric immersions from the GCR system, or equivalently the Cartan structural system) on simply-connected semi-Riemannian manifolds with lower regularity. Finally, in §6, we discuss further applications of the theorems and techniques established in earlier sections. In particular, we demonstrate the weak continuity of Einstein's constraint equations, quasilinear wave equations with the null structure, and general hypersurfaces in space-time. For completeness, the proofs of several semi-Riemannian geometric results and facts, as well as the proof of Theorem 3.5 (the generalized quadratic theorem on locally compact Abelian groups), are presented in Appendices A and B.

2 The Cartan Structural System and Isometric Immersions of Semi-Riemannian Manifolds

In this section, we discuss the Cartan structural system. One of our motivations comes from the isometric immersion problem for semi-Riemannian manifolds: the Cartan structural system is known to be equivalent to the GCR system, since both systems are the classical compatibility equations for the existence of isometric immersions. The isometric immersion problem is an important topic in theoretical physics and differential geometry. In particular, it is closely related to the definition of quasi-local mass in space-time (see Brown–York [6], Wang–Yau [62], and the references therein).

We first review the submanifold theory in semi-Riemannian geometry. Then we discuss the derivation of the GCR system and the formulation of the Cartan structural system. Our exposition follows essentially from O'Neill [51]; nevertheless, several *ad hoc* constructions therein are clarified by using the language of vector bundles.

2.1 Semi-Riemannian Submanifold Theory

Let M be an n-dimensional manifold. It is said to be *semi-Riemannian* if there exists a symmetric, non-degenerate 2-form field g on the tangent bundle TM with constant index. Then g is known as a *semi-Riemannian metric*. The semi-Riemannian metric g is *non-degenerate* on M if, for each $x \in M$, there exists no $v \in T_xM \setminus \{0\}$ such that $g(v,w) = 0$ for every $w \in T_xM$.

The *index* of the semi-Riemannian metric g on T_xM is defined by

$$\mathrm{Ind}(g; T_xM) := \max\left\{\dim V \,:\, \begin{array}{l} V \subset T_xM \text{ is a vector subspace} \\ \text{and } g|_V \text{ is negative definite} \end{array}\right\}.$$

Clearly, if M is connected, then $\mathrm{Ind}(g;T_xM)$ is constant for all $x\in M$, which will be written as $\mathrm{Ind}(g)$ in the sequel. Employing the Gram–Schmidt process to a subset $U\subset M$, we can find a local orthonormal basis $\{e_i\}_1^n\subset TU$ so that g is diagonalized:

$$g=\{g_{ij}\}=\delta_{ij}|g_{ij}|\epsilon^j \qquad \text{for each } i,j\in\{1,2,\dots,n\},$$

where $\epsilon:=(\epsilon^1,\dots,\epsilon^n)^\top\in\{-1,1\}^n$ is called the *signature* of metric g. As g is non-degenerate, it has only non-zero entries on the diagonal so that $\mathrm{Ind}(g)$ equals to the number of "-1" in signature ϵ. For simplicity, from now on, the semi-Riemannian manifold (M,g) is always taken to be connected, and $\mathrm{Ind}(g)$ is called the *index of* M for the fixed metric g.

Let $(\widetilde{M},\tilde g)$ be a given semi-Riemannian manifold, and let M be a submanifold via the embedding $\iota: M\hookrightarrow(\widetilde{M},\tilde g)$, *i.e.*, both $\iota: M\hookrightarrow(\widetilde{M},\tilde g)$ and $\mathrm{d}\iota: TM\to T\widetilde{M}$ are injective. We say that $(M,\iota^*\tilde g)$ is a *semi-Riemannian submanifold* of $(\widetilde{M},\tilde g)$, provided that $\iota^*\tilde g$ is non-degenerate on M, where $\iota^*\tilde g$ denotes the pullback of $\tilde g$ defined by

$$(\iota^*\tilde g)_x(v,w):=\tilde g_{\iota(x)}(\mathrm{d}\iota(v),\mathrm{d}\iota(w)) \qquad \text{for each } x\in M \text{ and } v,w\in T_xM.$$

Before further development, we introduce one notation: For any vector bundle E over M, we write $\Gamma(E)$ for the space of sections of E, *i.e.*, $s: M\to E$ such that $\pi\circ s=\mathrm{id}_M$, where $\pi: E\to M$ is the projection of bundle E onto the base manifold.

Next, we consider $\iota^*T\widetilde{M}$, the vector bundle with base manifold M and fiber $T_{\iota(x)}\widetilde{M}$ at each $x\in M$. Then $\Gamma(\iota^*T\widetilde{M})$ consists of the vector fields in $T\widetilde{M}$ defined along M. In particular,

$$\iota^*T\widetilde{M}|_x:=T_{\iota(x)}\widetilde{M}=\mathrm{d}_x\iota(T_xM)\oplus[\mathrm{d}_x\iota(T_xM)]^\perp\cong T_xM\oplus[\mathrm{d}_x\iota(T_xM)]^\perp, \tag{2.1}$$

whenever ι is a *local immersion*, *i.e.*, $\mathrm{d}\iota$ is injective in some neighborhood of $x\in M$. Here the direct sum is taken with respect to the bilinear form $\tilde g$ on $T_{\iota(x)}\widetilde{M}$:

$$\big[\mathrm{d}_x\iota(T_xM)\big]^\perp:=\big\{v\in T_{\iota(x)}\widetilde{M}\,:\,\tilde g_{\iota(x)}(v,\mathrm{d}_x\iota(w))=0\ \text{ for all } w\in T_xM\big\}.$$

Eq. (2.1) is a special case of Lemma 23 in [51], which is proved by a simple dimension-counting. It holds only when $\iota^*\tilde g$ is non-degenerate, *i.e.*, M is immersed into $(\widetilde{M},\tilde g)$ as a semi-Riemannian submanifold. In this case, TM and $\iota^*T\widetilde{M}$ are vector bundles over M and $TM\subset\iota^*T\widetilde{M}$ respectively, hence the quotient bundle is well-defined.

Definition 2.1 The normal bundle of the isometric immersion $\iota: M\hookrightarrow\widetilde{M}$ is

$$TM^\perp:=\frac{\iota^*T\widetilde{M}}{TM}.$$

In view of Eq. (2.1), the fiber of $TM^\perp$ at $x \in M$ (written as $T_xM^\perp$) is isomorphic to $[\mathrm{d}_x\iota(T_xM)]^\perp$, so that the following isomorphism of vector spaces holds:

$$T_{\iota(x)}\widetilde{M} \cong T_xM \oplus T_xM^\perp. \tag{2.2}$$

The canonical projections of $T_{\iota(x)}\widetilde{M}$ onto the first and second factors are called the *tangential* and *normal projections*, denoted by

$$\tan : \iota^* T\widetilde{M}|_x \to T_xM, \qquad \mathrm{nor} : \iota^* T\widetilde{M}|_x \to T_xM^\perp. \tag{2.3}$$

By naturality, they induce both the projections of vector fields:

$$\tan : \Gamma(\iota^* T\widetilde{M}) \to \Gamma(TM), \qquad \mathrm{nor} : \Gamma(\iota^* T\widetilde{M}) \to \Gamma(TM^\perp), \tag{2.4}$$

and the projections of vector fields with Sobolev regularity:

$$\begin{aligned}\tan &: W^{k,p}(M;\iota^* T\widetilde{M}) \to W^{k,p}(M;TM),\\ \mathrm{nor} &: W^{k,p}(M;\iota^* T\widetilde{M}) \to W^{k,p}(M;TM^\perp)\end{aligned}$$

for $p \in [1,\infty]$ and $k \in \mathbb{Z}$.

Moreover, for notational convenience, we introduce the following conventions:

Convention 2.2 *We write the tangential vector fields as $X, Y, Z, \ldots \in \Gamma(TM)$ and the normal vector fields as $\xi, \eta, \zeta, \ldots \in \Gamma(TM^\perp)$. For a generic vector field not necessarily tangential or normal, i.e., an element in $\Gamma(T\widetilde{M})$ or $\Gamma(\iota^* T\widetilde{M})$, we use letters U, V, W. Finally, for a bundle E different from TM, $TM^\perp$, and $\iota^* T\widetilde{M}$, we write $\alpha, \beta, \ldots \in \Gamma(E)$.*

Convention 2.3 *Given an isometric immersion $f : M \to \widetilde{M}$, write $\{\partial_a\}$, S, g, ∇, R, $\ldots$ for the geometric quantities on M, and $\{\widetilde{\partial}_a\}, \widetilde{S}, \tilde{g}_0, \widetilde{\nabla}, \widetilde{R}, \ldots$ for the corresponding quantities on $\widetilde{M}$.*

With the orthogonal splitting of tangent and normal directions under isometric immersions, we are ready to study the orthogonal splitting of connections. Let $(\widetilde{M}, \tilde{g})$ be a semi-Riemannian manifold, and let $\iota : M \hookrightarrow \widetilde{M}$ be an immersed semi-Riemannian submanifold. The Levi–Civita theorem says that there exists a unique affine connection $\widetilde{\nabla} : \Gamma(T\widetilde{M}) \times \Gamma(T\widetilde{M}) \to \Gamma(T\widetilde{M})$ which is metric-compatible and torsion-free (*cf.* [51]). More precisely, the following conditions hold for any smooth function $\varphi : M \to \mathbb{R}$ and vector fields $U, V, W \in \Gamma(T\widetilde{M})$:

(i) Affine: $\widetilde{\nabla}_{\varphi V} W = \varphi \widetilde{\nabla}_V W$ and $\widetilde{\nabla}_V(\varphi W) = V(\varphi)W + \varphi \widetilde{\nabla}_V W$;

(ii) Compatible with metric: $U\tilde{g}(V,W) = \tilde{g}(\widetilde{\nabla}_U V, W) + \tilde{g}(V, \widetilde{\nabla}_U W)$;

(iii) Torsion-free: $\widetilde{\nabla}_V W - \widetilde{\nabla}_W V = [V, W]$.

Recall that the connections can be pulled back by using the maps between topological manifolds (see *e.g.* [56]). In particular, $\iota : M \hookrightarrow \widetilde{M}$ induces the *pullback connection* $\iota^*\widetilde{\nabla} : \Gamma(TM) \times \Gamma(\iota^* T\widetilde{M}) \to \Gamma(\iota^* T\widetilde{M})$ on the pullback bundle $\iota^* T\widetilde{M}$, given by

$$(\iota^*\widetilde{\nabla})_X(\iota^*\alpha) = \iota^*(\widetilde{\nabla}_{\mathrm{d}\iota(X)}\alpha) \qquad \text{for any } \alpha \in \Gamma(T\widetilde{M}) \text{ and } X \in \Gamma(TM).$$

Hence, for a vector field $V \in \Gamma(T\widetilde{M})$ along M, *i.e.*, $V \in \Gamma(\iota^* T\widetilde{M})$, we have

$$(\iota^*\widetilde{\nabla})_X V = \iota^*(\widetilde{\nabla}_{\mathrm{d}\iota(X)}\mathrm{d}\iota(V)) = \widetilde{\nabla}_{\mathrm{d}\iota(X)}\mathrm{d}\iota(V), \tag{2.5}$$

where $\mathrm{d}\iota(X)$ and $\mathrm{d}\iota(V)$ can be viewed as the local extensions of $X \in \Gamma(TM)$ and $V \in \Gamma(\iota^* T\widetilde{M})$ to the vector fields in $\Gamma(T\widetilde{M})$.

For simplicity, we adopt the slight abuse of notations of systematically dropping the pullback operator ι^* (see [26, 51, 61]) when no confusion arises. In effect, this amounts to viewing M as a subset of $\widetilde{M}$, and ι as the identity map from M to its image.

Convention 2.4 *Let $\iota : (M, g) \hookrightarrow (\widetilde{M}, \tilde{g})$ be an isometric immersion of semi-Riemannian submanifolds. Then $(\iota^* T\widetilde{M}, \iota^*\widetilde{\nabla})$ is replaced by $(T\widetilde{M}, \widetilde{\nabla})$.*

With the above preparations, we now consider the following decomposition of connections:

$$\widetilde{\nabla}_X V = \tan[\widetilde{\nabla}_X(\tan V)] + \tan[\widetilde{\nabla}_X(\mathrm{nor} V)] + \mathrm{nor}[\widetilde{\nabla}_X(\tan V)] + \mathrm{nor}[\widetilde{\nabla}_X(\mathrm{nor} V)]$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T\widetilde{M})$, where both projections *tan* and *nor* are as in Eq. (2.4).

Definition 2.5 Given an isometric immersion $\iota : (M, g) \hookrightarrow (\widetilde{M}, \tilde{g})$, the tangential connection $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$, the second fundamental form $\mathrm{II} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$, the shape operator (associated to II) $S : \Gamma(TM) \times \Gamma(TM^\perp) \to \Gamma(TM)$, and the normal connection $\nabla^\perp : \Gamma(TM) \times \Gamma(TM^\perp) \to \Gamma(TM^\perp)$ are defined as

$$\begin{cases} \nabla_X Y := \tan \widetilde{\nabla}_X Y, & \mathrm{II}(X, Y) := \mathrm{nor}\, \widetilde{\nabla}_X Y, \\ S_\xi X := -\tan \widetilde{\nabla}_X \xi, & \nabla^\perp_X \xi := \mathrm{nor}\, \widetilde{\nabla}_X \xi, \end{cases}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$.

We note that ∇ is the Levi–Civita connection on $(M, \iota^*\tilde{g})$, whenever $\widetilde{\nabla}$ is the Levi–Civita connection on $\widetilde{M}$. Moreover, II and S are related by

$$\tilde{g}(\mathrm{II}(X, Y), \xi) = \tilde{g}(S_\xi X, Y).$$

In addition, II is symmetric (equivalently, S_ξ is self-adjoint) on $\Gamma(TM)$. The Riemann curvature tensor will be introduced in §2.2 below.

Finally, with $\mathfrak{gl}(n;\mathbb{R})$ denoting the space of $n\times n$ real matrices, we define the *semi-orthogonal group of* $\mathbb{R}^n_\nu$ as

$$O(\nu, n-\nu) := \Big\{B \in \mathfrak{gl}(n;\mathbb{R}) : B(v,w) = \epsilon_{n,\nu} v\cdot w \text{ for all } v,w \in T\mathbb{R}^n_\nu\Big\},$$

with the signature matrix given by

$$\epsilon_{n,\nu} = \operatorname{diag}(\underbrace{-1,\cdots,-1}_{\nu \text{ times}}, \underbrace{1,\cdots,1}_{n-\nu \text{ times}}). \tag{2.6}$$

In other words, $O(\nu, n-\nu)$ is the group of linear isometries from $\mathbb{R}^n_\nu$ to itself. Here and in the sequel, $\mathbb{R}^n_\nu$ denotes the *semi-Euclidean* space, *i.e.*, manifold $\mathbb{R}^n$ equipped with metric $\epsilon_{n,\nu}$. Likewise, the Lie group $O(\tau, k-\tau)$ has the signature matrix:

$$\epsilon_{k,\tau} = \operatorname{diag}(\underbrace{-1,\cdots,-1}_{\tau \text{ times}}, \underbrace{1,\cdots,1}_{k-\tau \text{ times}}).$$

We also denote by $\mathbb{R}^{n+k}_{\nu+\tau}$ the semi-Euclidean space $\mathbb{R}^{n+k}$ with the metric:

$$\tilde{g}_0 = \epsilon_{n,\nu} \oplus \epsilon_{k,\tau}.$$

The direct sum is understood as the block sum of matrices. Furthermore, we denote the Lie algebra of $O(n, n-\nu)$ as $\mathfrak{o}(n, n-\nu)$.

2.2 Gauss–Codazzi–Ricci System and Isometric Immersions

The isometric immersion problem can be stated as follows: *Given a semi-Riemannian manifold (M,g) and a target semi-Riemannian manifold $(\widetilde{M},\tilde{g})$ of higher dimension, seek an immersion $f:(M,g)\hookrightarrow(\widetilde{M},\tilde{g})$ such that $f(M)$ is a semi-Riemannian submanifold of $\widetilde{M}$ with $f^*\tilde{g}=g$.*

A necessary compatibility condition for the existence of an isometric immersion f is that the Riemann curvature tensor of $\widetilde{M}$ should be splitted nicely in the tangential and normal directions, *i.e.*, in TM and $TM^\perp$. In what follows, we discuss the Riemann curvature on semi-Riemannian manifolds and derive the compatibility equations, which are known as the GCR system. Again, for our purpose, we focus on the perspectives of vector bundles, in comparison with [51]. One further convention is introduced for notational convenience:

Convention 2.6 *In the rest of the paper, we write $\langle\cdot,\cdot\rangle$ for $\tilde{g}(\cdot,\cdot), g(\cdot,\cdot)$, and any other semi-Riemannian metrics, unless further specified.*

Let (M,g) be an n-dimensional semi-Riemannian manifold of index ν, and let E be a vector bundle over M with fibers $F\cong\mathbb{R}^k_\tau$, the *semi-Euclidean space $\mathbb{R}^k$ with index τ*. Let ∇^E be an affine connection on bundle E, *i.e.*, a linear map

$$\nabla^E : \Gamma(TM)\times\Gamma(E)\to\Gamma(E)$$

satisfying $\nabla^E_{\phi X}\alpha = \phi\nabla^E_X\alpha$ and $\nabla^E_X(\phi\alpha) = X(\phi)\alpha + \phi\nabla^E_X\alpha$ for any $\phi : M \to \mathbb{R}$. This can be compactly written as

$$\nabla^E(\phi\alpha) = \phi\nabla^E\alpha + \mathrm{d}\phi \otimes \alpha,$$

once we view $\nabla^E : \Gamma(E) \to \Omega^1(E) := \Gamma(E \otimes T^*M)$, the space of differential 1-forms on bundle E. The *Riemann curvature on bundle* E is given by $R^E : \Gamma(TM) \times \Gamma(TM) \to \Gamma(\mathrm{End}E)$ as

$$R^E(X,Y) := [\nabla^E_X, \nabla^E_Y] - \nabla^E_{[X,Y]},$$

where $\mathrm{End}E$ is the *endomorphism bundle* on E. That is, $\mathrm{End}E$ is the vector bundle over M with the typical fiber $\mathfrak{gl}(F)$, the group of linear transforms from F to itself. Note that $R^E(X,Y,\alpha) \in \Gamma(E)$ for $\alpha \in \Gamma(E)$. Also, R^E is often written as the $(0,4)$–tensor:

$$R^E(X,Y,\alpha,\beta) := \langle R^E(X,Y,\alpha),\beta\rangle_E \qquad \text{for } X,Y \in \Gamma(TM) \text{ and } \alpha,\beta \in \Gamma(E),$$

where we write $\langle\cdot,\cdot\rangle_E$ to emphasize the bundle metric.

Now we may investigate the orthogonal splitting of the Riemann curvature along the projections *tan* and *nor* (see §2.1). Given an isometric immersion $f : (M,g) \to (\widetilde{M},\tilde{g})$, three vector bundles over M are of interest: $E = TM$, $TM^\perp$, and $f^*T\widetilde{M}$. We denote the last bundle by $T\widetilde{M}$ in light of Convention 2.2. We also fix the notations:

$$\begin{cases} \nabla = \nabla^{TM}, & \widetilde{\nabla} = \nabla^{T\widetilde{M}}, \qquad \nabla^\perp = \nabla^{TM^\perp}, \\ R = R^{TM}, & \widetilde{R} = R^{T\widetilde{M}}, \qquad R^\perp = R^{TM^\perp}, \end{cases}$$

where ∇^{TM} denotes the Levi–Civita connection on M.

In what follows, we are concerned with the special case:

$$\widetilde{M} = \mathbb{R}^{n+k}_{\nu+\tau}, \qquad \mathrm{Ind}(\widetilde{M}) = \mathrm{Ind}(M) + \mathrm{Ind}(\mathbb{R}^k_\tau).$$

Thus, $\widetilde{R}(X,Y) \in \Gamma(\mathrm{End}T\widetilde{M})$ constantly vanishes so that

$$\widetilde{R}(X,Y,Z_1,Z_2) = 0, \qquad \widetilde{R}(X,Y,Z,\xi) = 0, \qquad \widetilde{R}(X,Y,\xi,\eta) = 0, \tag{2.7}$$

for arbitrary $Z, Z_1, Z_2 \in \Gamma(TM)$ and $\xi,\eta \in \Gamma(TM^\perp)$. Applying projections *tan* and *nor* to Eq. (2.7) and expressing them via $R, R^\perp, \mathrm{II}, S$, and ∇ as in Definition 2.5, we deduce

Theorem 2.1 *The following three equations are equivalent to Eq.* (2.7):

$$\langle \mathrm{II}(X,Z_1),\mathrm{II}(Y,Z_2)\rangle - \langle \mathrm{II}(X,Z_2),\mathrm{II}(Y,Z_1)\rangle = R(X,Y,Z_1,Z_2), \tag{2.8}$$

$$\nabla^\perp_X \mathrm{II}(Y,Z) = \nabla^\perp_Y \mathrm{II}(X,Z), \tag{2.9}$$

$$\langle [S_\xi, S_\eta]X, Y\rangle = -R^\perp(X,Y,\xi,\eta) \tag{2.10}$$

for any $X,Y,Z_1,Z_2 \in \Gamma(TM)$ *and* $\xi,\eta \in \Gamma(TM^\perp)$, *where the covariant derivative of* II *is defined via the Leibniz rule*:

$$\nabla^\perp_X \mathrm{II}(Y,Z) = X(\mathrm{II}(Y,Z)) - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(Y, \nabla_X Z).$$

A sketched proof of the above theorem is given in Appendix A.1, which is analogous to the derivation in do Carmo [26, §6] for the Riemannian case. The three equations (2.8), (2.9), and (2.10) are named after Gauss, Codazzi, and Ricci, respectively, which form the GCR system.

Three remarks on the GCR system are in order:

(i) The GCR system is a first-order nonlinear PDE system on the semi-Riemannian manifold (M, g), with given g (hence ∇ and R) and unknowns $(\mathrm{II}, \nabla^{\perp})$. The nonlinear terms in this system are of forms $\mathrm{II} \otimes \mathrm{II}$, $\mathrm{II} \otimes \nabla^{\perp}$, or $\nabla^{\perp} \otimes \nabla^{\perp}$, which are of *quadratic nonlinearity.*

(ii) The GCR system in Theorem 2.1 takes the same form as in the Riemannian case*;* see [12,26,55]. Such coincidence, nevertheless, is merely formal. The GCR system for semi-Riemannian manifolds includes the information of non-trivial signatures, which leads to further analytical difficulties.

(iii) The GCR system can be generalized to any vector bundle E in place of $TM^{\perp}$. Indeed, since the Riemann curvature is defined for any bundle E (*i.e.*, R^E), for any symmetric tensor $\mathrm{II} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(E)$ and $S : \Gamma(E) \times \Gamma(TM) \to \Gamma(TM)$ given by

$$\langle S_\alpha X, Y \rangle = \langle \mathrm{II}(X, Y), \alpha \rangle \qquad \text{for } X, Y \in \Gamma(TM) \text{ and } \alpha \in \Gamma(E),$$

the GCR system in Theorem 2.1 is still well-defined for $X, Y, Z_1, Z_2 \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(E)$, wherein we replace $R^{\perp}$ by R^E in Eq. (2.10). Such equations are called *the GCR system on bundle E*.
Suppose that the trivial bundle of the ambient semi-Euclidean space $T\mathbb{R}^{n+k}$ admits an orthogonal splitting $TM \oplus E$ as the Whitney sum of vector bundles. Then it is clear that the GCR system on bundle E is necessary for the splitting. Conversely, we will prove in Theorem 5.1 that, for an abstract vector bundle E over M, the GCR system on E is also a sufficient condition for the local existence of such a splitting. Moreover, the splitting holds globally if M is simply-connected, under suitable regularity assumptions.

2.3 Cartan Structural System

Now we introduce the *Cartan structural system* for the semi-Riemannian submanifolds, first appeared in the formalism of exterior differential calculus due to E. Cartan (*cf.* [20]). This can be viewed as an equivalent form of the Gauss–Codazzi–Ricci system, which is more suitable for the weak continuity and realizability considerations in the subsequent sections.

Cartan's formalism (*a.k.a.* the method of moving frames) is a classical tool in differential geometry; see [15, 55, 57]. In particular, it plays a crucial role in the establishment of the realization theorem for Riemannian submanifolds by Tenenblat [61], as well as the existence and uniqueness of immersions of smooth manifolds into affine homogeneous spaces by Eschenburg–Tribuzy [27]. In this paper, we develop Cartan's formalism for the semi-Riemannian submanifolds. It serves as the foundation for the Cartan structural system.

To set up Cartan's formalism, we need to introduce the frame field on TM and its co-frame field on T^*M, as well as the field of connection 1-forms. The following convention is adopted:

Convention 2.7 *From now on, the superscripts and subscripts obey the following rule*:

$$1 \leq i,j,k,l,s,t \leq n; \qquad n+1 \leq \alpha,\beta,\gamma \leq n+k; \qquad 1 \leq a,b,c,e \leq n+k.$$

Now, let $\{\partial_1, \ldots, \partial_n\} \subset \Gamma(TM)$ be a frame field for M; that is, at each point P on M, $\{\partial_i|_P\}_1^n$ forms an orthonormal basis for the tangent space T_PM. The orthonormality means

$$\langle \partial_i, \partial_j \rangle = \delta^i_j \epsilon^i \qquad \text{for all } i,j \in \{1,\ldots,n\}$$

in the semi-Riemannian settings. We write $\{\theta^1,\ldots,\theta^n\} \subset \Gamma(T^*M)$ for the co-frame field:

$$\theta^i(\partial_j) = \delta^i_j.$$

Similarly, we can also take $\{\partial_{n+1},\ldots,\partial_{n+k}\} \subset \Gamma(E)$ to be a frame field for E, *i.e.*, orthonormal with respect to the bundle metric g^E, and $\{\theta^{n+1},\ldots,\theta^{n+k}\} \subset \Gamma(E^*)$ to be its co-frame field.

In light of Convention 2.7, we define the *connection* 1*-forms*:

Definition 2.8 Let (M,g) be a semi-Riemannian manifold, and let E be a vector bundle over M with bundle metric g^E. The connection 1-form $\mathcal{W}$ is a 1-form-valued $(n+k)\times(n+k)$ matrix field:

$$\mathcal{W} = \{\omega^a_b\} \in \Gamma(\mathfrak{gl}(n+k;\mathbb{R}) \otimes T^*M),$$

defined component-wise as

$$\begin{cases} \omega^i_j(\partial_l) := \theta^j(\nabla_{\partial_l}\partial_i) = \epsilon^j \langle \nabla_{\partial_l}\partial_i, \partial_j\rangle, \\ \omega^i_\alpha(\partial_j) := \theta^\alpha(\mathrm{II}(\partial_i,\partial_j)) = \epsilon^\alpha \langle \mathrm{II}(\partial_i,\partial_j), \partial_\alpha\rangle, \\ \omega^\alpha_\beta(\partial_i) := \theta^\beta(\nabla^E_{\partial_i}\partial_\alpha) = \epsilon^\beta \langle \nabla^E_{\partial_i}\partial_\alpha, \partial_\beta\rangle, \\ \omega^\alpha_i := -\epsilon^i\epsilon^\alpha\omega^i_\alpha. \end{cases} \tag{2.11}$$

Remark 2.1 We identify

$$\Gamma(\mathfrak{gl}(n+k;\mathbb{R}) \otimes T^*M) \cong \Gamma(\mathfrak{gl}(TM \oplus E) \otimes T^*M) =: \Omega^1(\mathfrak{gl}(n+k;\mathbb{R})).$$

The right-most expression means the space of $\mathfrak{gl}(n+k;\mathbb{R})$-valued differential 1-forms. In general, for a Lie algebra $\mathfrak{g}$, the space of differential k-forms with entries in $\mathfrak{g}$ is written as

$$\Omega^k(\mathfrak{g}) := \Gamma(\wedge^k T^*M \otimes \mathfrak{g}). \tag{2.12}$$

This notation is needed for subsequent development.

Now we introduce the two Cartan structural systems for semi-Riemannian manifolds, the second of which is equivalent to the GCR system introduced in §2.2. This seems to be known in the semi-Riemannian geometry community; nevertheless, we have not been able to locate a proof in the literature, so it is needed to present a detailed proof for completeness in Appendix A.3.

Proposition 2.1 *The GCR system* (2.8)–(2.10) *is equivalent to the following system for the connection* 1*-form* (*known as the second structural system*):

$$\mathrm{d}\mathcal{W} = \mathcal{W} \wedge \mathcal{W}. \tag{2.13}$$

Its proof relies on a key lemma (see Appendix A.2), which says that $\mathcal{W}$ is a "semi-skew-symmetric" matrix:

Lemma 2.1 $\mathcal{W} = \{\omega^a_b\} \in \Omega^1(\mathfrak{o}(\nu+\tau; (n+k)-(\nu+\tau)))$.

For subsequent developments, we note that $\mathcal{W}$ can be schematically represented in the block-matrix form:

$$\{\omega^b_a\}_{1\le a,b\le k+n} = \begin{bmatrix} \omega^i_j & \omega^\alpha_i \\ \omega^i_\alpha & \omega^\beta_\alpha \end{bmatrix} = \begin{bmatrix} \theta^j(\nabla_\bullet \partial_i) & S_{\partial_\alpha}\partial_i \\ -S_{\partial_\alpha}\partial_i & \theta^\beta(\nabla^E_\bullet \partial_\alpha) \end{bmatrix}. \tag{2.14}$$

Remark 2.2 System (2.13) is understood *as an equality on* $\Omega^2(\mathfrak{g})$. On the left-hand side, the exterior differential d is viewed as acting only on the T^*M factor if $\mathcal{W} \in \Omega^1(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{gl}(n+k;\mathbb{R})$ in Eq. (2.12). Then $\mathrm{d}\mathcal{W} \in \Omega^2(\mathfrak{g})$ and is given by

$$\mathrm{d}\mathcal{W}(U,V) := U(\mathcal{W}(V)) - V(\mathcal{W}(U)) + \mathcal{W}([U,V]) \qquad \text{for all } U,V \in \Gamma(TM).$$

On the right-hand side, the wedge product on $\Omega^1(\mathfrak{g})$ is taken by combining the wedge product on the T^*M factor and the matrix multiplication on the $\mathfrak{g}$ factor in Eq. (2.12). That is,

$$(\mathcal{W}\wedge\mathcal{W})(U,V) := \mathcal{W}(U)\cdot\mathcal{W}(V) - \mathcal{W}(V)\cdot\mathcal{W}(U) \qquad \text{for all } U,V \in \Gamma(TM).$$

So far, we have established the equivalence between the GCR system and system (2.13). It is known as the *second* structural system. In fact, the *first* structural system consists of the following identities on $\Omega^1(\mathfrak{gl}(n;\mathbb{R}))$:

$$\mathrm{d}\theta = \theta \wedge \mathcal{W}. \tag{2.15}$$

This is equivalent to the torsion-free property of connection ∇. As this property is independent of metrics (regardless of Riemannian or semi-Riemannian), it does not provide additional information to the isometric immersions. The proof is standard and is sketched in Appendix A.4.

In the rest of the paper, we always refer to the second structural system (2.13) as the Cartan structural system. In §4, we establish its global weak continuity.

3 Weak Continuity of Quadratic Functions on Semi-Riemannian Manifolds

In order to establish the weak continuity of the Cartan structural system on semi-Riemannian manifolds with lower regularity, we need to pass to the weak limit of the quadratic nonlinear term $\mathcal{W}\wedge\mathcal{W}$, where $\mathcal{W}$ is the connection 1-form in Proposition 2.1. We establish a geometrically intrinsic compensated compactness theorem on vector bundles over the semi-Riemannian manifold and apply it to develop a geometric, global approach to our problem. This is the main goal of this section.

Our generalized quadratic theorem concerns the weakly convergent L^2 sections of a vector bundle E over a semi-Riemannian manifold M. Its prototype is the quadratic theorem *à la* Tartar [59] on the Euclidean space $\mathbb{R}^n$. In order to formulate it globally and intrinsically, two difficulties immediately arise:

(i) Being endowed with a semi-Riemannian metric, M is a *real* manifold. However, our proof is based on Fourier analysis below involving factor $i=\sqrt{-1}$, which has to be carried out over $\mathbb{C}$.

(ii) The Fourier transform cannot be defined globally on a generic semi-Riemannian manifold. For $u\in L^2(M;E)$, one way we can do is to define

$$\hat{u}(x,\xi):=\int_M u(y)e^{-2\pi i\langle \exp_x^{-1}(y),\xi\rangle}\,\mathrm{d}V_g(y)\qquad \text{for } x\in M, \xi\in T_x^*M,$$

where $\exp_x: T_xM\to M$ is the exponential map on the manifold, $\langle\cdot,\cdot\rangle$ is the paring of TM and T^*M given by metric g, and $\mathrm{d}V_g$ is the volume form of g. However, it is only well-defined at $x\in M$ up to the first conjugate point of x, for which $\exp_x^{-1}$ can be specified unambiguously.

The above considerations call for a quadratic theorem on real manifolds, for which the differential constraints are formulated globally and intrinsically. For this purpose, we introduce three new ingredients:

- The (principal) symbol of a differential operator,
- A quadratic polynomial defined globally on vector bundles,
- Complexifications of vector bundles and quadratic polynomials.

The rest of this section is organized as follows: We first present the definitions and basic properties of the principal symbol, quadratic polynomials, and the Sobolev norms of sections over semi-Riemannian manifolds. Then our generalized quadratic theorem is first stated and proved over a semi-Riemannian manifold with a C^∞ metric (*cf.* Theorem 3.1) and is then extended over a semi-Riemannian manifold with a non-degenerate L^∞ metric (*cf.* Theorem 3.2). From now on, let M be a semi-Riemannian manifold, and let E and F be two real vector bundles over M.

Principal Symbols. We collect only some basic facts here, and refer to [1] for the details.

Denote $\mathcal{T} \in \mathrm{Diff}^m(M; E, F)$ as an arbitrary differential operator $\mathcal{T}$ of order m that maps E-sections to F-sections:

$$\mathcal{T} : \Gamma(E) \to \Gamma(F).$$

It is a crucial observation in micro-local analysis that $\sigma_m(\mathcal{T})$, the *principal symbol* of $\mathcal{T}$, can be defined intrinsically. Indeed, for any $\xi \in T_x^*M$, we may choose a function $f \in C^\infty(M)$ such that $\mathrm{d}_x f = \xi$, and then set

$$\sigma_m(\mathcal{T})(x,\xi) := \lim_{t\to\infty} \frac{[e^{-2\pi itf} \circ \mathcal{T} \circ e^{2\pi itf}](x)}{t^m}. \tag{3.1}$$

It is easy to check that $\sigma_m(\mathcal{T})(x,\xi) \in \mathrm{Hom}(E_x, F_x)$ for any given ξ and that the definition is independent of the choice of f. Here and hereafter, $E_x \cong \mathbb{R}^J$ and $F_x \cong \mathbb{R}^I$ denote the fiber of E and F at point $x \in M$, respectively, and $\mathrm{Hom}(E_x, F_x)$ denotes the space of vector space homomorphisms from E_x to F_x. Moreover, σ_m is a homogeneous polynomial of order m on each fiber of T^*M:

$$\sigma_m(\mathcal{T})(x,\lambda\xi) = |\lambda|^m \sigma_m(\mathcal{T})(x,\xi) \qquad \text{for all } x \in M, \xi \in T_x^*M,\ \lambda \in \mathbb{C}.$$

More abstractly, denoting $\mathcal{P}_l(V,W)$ as the vector space of l-degree homogeneous polynomials between the vector bundles V and W, the principal symbol map σ_m defines the following vector space homomorphism:

$$\sigma_m : \mathrm{Diff}^m(M; E, F) \to \mathcal{P}_m(T^*M; \mathrm{Hom}(E; F^{\mathbb{C}})),$$

where $F^{\mathbb{C}} := F \otimes_{\mathbb{R}} \mathbb{C}$ is the complexified vector bundle, which is necessary since $i = \sqrt{-1}$ appears in the definition of σ_m in Eq. (3.1). We adopt this abstract language in order to emphasize the global, intrinsic nature of the principal symbol.

For the application in §5, we now discuss the following example: The exterior differential operator $\mathcal{T} = \mathrm{d} : \wedge^q T^*M \to \wedge^{q+1} T^*M$. In fact, we have

$$\mathrm{d} \in \mathrm{Diff}^1(M; \wedge^q T^*M, \wedge^{q+1} T^*M),$$

whose the principal symbol $\sigma_1(\mathrm{d})$ is given by

$$[\sigma_1(\mathrm{d})(\xi)](\omega) = -2\pi i\xi \wedge \omega \qquad \text{for } \xi \in T^*M \text{ and } \omega \in \wedge^q T^*M.$$

Owing to the presence of $i = \sqrt{-1}$, we view the exterior algebra in the range of d as being complexified: For each $\xi \in T^*M$, $\sigma_1(\mathrm{d})(\xi) \in \mathcal{P}_1(\wedge^q T^*M;\ \wedge^{q+1} T^*M \otimes \mathbb{C})$. In this case, notice that $\sigma_1(\mathrm{d})(\xi) = -2\pi i\xi\wedge$, which is indeed a 1-homogeneous polynomial of operators from q-tensors to complexified $(q+1)$-tensors.

Intrinsic Formulation of Quadratic Polynomials. Now we define a quadratic polynomial on a vector bundle E:

Definition 3.1 Let E be a vector bundle over a real manifold M. A map $Q : \Gamma(E) \to \mathbb{C}$ is a quadratic polynomial on E if it factors as

$$Q : \Gamma(E) \xrightarrow{j} \Gamma(E \otimes E) \xrightarrow{\mathbf{q}} \mathbb{C},$$

where $j(s) = (s, s)$ is the natural inclusion of the diagonal, and $\mathbf{q} \in \Gamma(\mathrm{Hom}(E \otimes E; \mathbb{C}))$ is conjugate 1-homogeneous in each argument:

$$\mathbf{q}(\lambda s_1, s_2) = \lambda \mathbf{q}(s_1, s_2), \qquad \mathbf{q}(s_1, \mu s_2) = \overline{\mu} \mathbf{q}(s_1, s_2)$$

for all $s_1, s_2 \in \Gamma(E)$ and $\lambda, \mu \in \mathbb{C}$. In this case, we write $Q \in \mathcal{P}_2(E; \mathbb{C})$.

Such constructions remain valid for $\mathbb{C}$ replaced by $\mathbb{R}$, in which Q is said to be a real quadratic polynomial on E. It follows from the definition that any quadratic polynomial Q is 2-homogeneous:

$$Q(\lambda s) = |\lambda|^2 Q(s) \qquad \text{for all } s \in \Gamma(E) \text{ and } \lambda \in \mathbb{C}.$$

Moreover, suppose that $U \subset M$ is a trivialized chart for the vector bundle E of degree J, *i.e.*, there exists a diffeomorphism:

$$\Phi : E \supset \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^J.$$

Then, for $s = \Phi^{-1}(x, z) \in U \times \mathbb{C}^J$ with $(x, z) \in U \times \mathbb{C}^J$, the value of the quadratic polynomial Q at s is given by

$$Q(s) = \sum_{j,k=1}^{J} Q_{jk}(x) z^j \overline{z^k} \qquad \text{with } Q_{jk} \in C^\infty(U), \tag{3.2}$$

so that the local representation of Q is obtained.

Sobolev Norms over Semi-Riemannian Manifolds. Now let us explain the construction of Sobolev norms (of sections of vector bundles) over semi-Riemannian manifolds.

Let (M, g) be a semi-Riemannian manifold. As we are concerned only with the local Sobolev spaces over M in this paper (see Theorems 3.1, 4.1, and 5.2), without loss of generality, we may assume M to be compact. Let $\mathfrak{U} := \{U_j\}_{j=1}^J$ be an atlas of coordinate charts on M. Given an arbitrary (r, s)-tensor field $\mathbf{T}$ on (M, g), by restricting to each chart in $\mathfrak{U}$, one may express it in local coordinates by $\mathbf{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}$. More precisely, let $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ be an local orthonormal basis for g, *i.e.*, $g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \epsilon^j \delta_{ij}$ (no summation) with $\epsilon = (\epsilon^1, \ldots, \epsilon^n)$ as the signature of g, and let $\{\mathrm{d}x^i\}$ be the co-frame dual to $\{\frac{\partial}{\partial x^i}\}$ via g. Then

$$\mathbf{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} := \mathbf{T}\big(\frac{\partial}{\partial x^{j_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{j_s}} \otimes \mathrm{d}x^{i_1} \otimes \ldots \otimes \mathrm{d}x^{i_r}\big).$$

The inner product of two (r, s)-tensor fields $\mathbf{T}$ and $\mathbf{S}$ on (M, g) is given by

$$\langle \mathbf{T}, \mathbf{S} \rangle_g := \mathbf{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \mathbf{S}^{a_1,\ldots,a_r}_{b_1,\ldots,b_s} g_{i_1,a_1} \cdots g_{i_r,a_r} g^{j_1,b_1} \cdots g^{j_s,b_s}, \tag{3.3}$$

where $\mathbf{T} = \{\mathbf{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}\}$ and $\mathbf{S} = \{\mathbf{S}^{a_1,\ldots,a_r}_{b_1,\ldots,b_s}\}$ in the local coordinates of $\mathfrak{U}$. Then we set

$$|\mathbf{T}|_g := \sqrt{|\langle \mathbf{T}, \mathbf{T}\rangle_g|}\,.$$

Note that Eq. (3.3) can be readily interpreted as an $L^1_{\rm loc}$ function when g is invertible *a.e.*, g_{ij} lies in L^∞ for each i,j, and $\mathbf{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}$ and $\mathbf{S}^{a_1,\ldots,a_r}_{b_1,\ldots,b_s}$ lie in L^p, $p \geq 2$, for all possible indices $i_1,\ldots,i_r, j_1,\ldots,j_s, a_1,\ldots,a_r$, and $b_1,\ldots,b_s$.

Now, take a scalar function $f:(M,g)\to\mathbb{R}$. Similar to the Riemannian case (*cf.* Chapter 2 in Hebey [35]), we define its $W^{k,p}$ norm, $k=0,1,2,\ldots$, by

$$\|f\|_{W^{k,p}(M,g)} := \begin{cases} \sum_{m=0}^k \Big\{ \int_M \big(|\nabla^m f|_g\big)^p \,\mathrm{d}V_g \Big\}^{\frac{1}{p}} & \text{for } p\in[1,\infty), \\ \sum_{m=0}^k \operatorname{ess\,sup}_M |\nabla^m f|_g & \text{for } p=\infty. \end{cases} \tag{3.4}$$

In the above, $\nabla^m := \overbrace{\nabla\circ\ldots\circ\nabla}^{m \text{ times}}$ denotes the iterated covariant derivatives, and the semi-Riemannian volume form is

$$\mathrm{d}V_g := \sqrt{|\det\, g|}\,\mathrm{d}\mathcal{L}^n \tag{3.5}$$

on each local chart of $\mathfrak{U}$, with the Lebesgue measure $\mathcal{L}^n$. The integration of a scalar function on M with respect to $\mathrm{d}V_g$ is defined in the standard way, by using an arbitrary partition of unity subordinate to $\mathfrak{U}$. The Sobolev space $W^{k,p}(M,g)$ is the completion of $C^\infty(M)$ under the norm in Eq. (3.4). For $k<0$ and $p\in[1,\infty]$, $W^{k,p}(M,g)$ is defined as the dual space of $W^{-k,p'}(M,g)$, where $\frac{1}{p}+\frac{1}{p'}=1$.

A tensor field $\mathbf{T}$ on (M,g) is said to have $W^{k,p}$–regularity if and only if $\mathbf{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} \in W^{k,p}(M,g)$ for all indices $i_1,\ldots,i_r$, and $j_1,\ldots,j_s$. Similarly, a connection on TM is $W^{k,p}$ if and only if its Christoffel symbols $\Gamma^\alpha_{\beta\gamma} \in W^{k,p}(M,g)$ for all α,β, and γ. Given a vector bundle E over (M,g) equipped with the bundle metric g^E, we write $W^{k,p}(M,g;E,g^E)$ for the space of E-sections with $W^{k,p}$ regularity, defined in an analogous manner as for tensor fields by considering trivialized charts for E.

We remark that the above definition of the $W^{k,p}$–norms may depend on the atlas $\mathfrak{U}$ and the trivialization of bundle E. Nonetheless, all these norms are equivalent modulo constants depending only on the differentiable structure of M. Thus, the corresponding Sobolev spaces are identical vector spaces with equivalent topologies; in particular, they are independent of local coordinates.

From now on, we assume that the semi-Riemannian metric g lies in $L^\infty_{\rm loc}$, with the non-degeneracy condition (see §2.1) understood in the *a.e.* sense. This is a very natural and mild condition, which suggests that M as a metric space does not contain interior infinity points. As a consequence, $\det g$ and g^{-1} (*e.g.*, obtained from the Cramer's rule) are also in $L^\infty_{\rm loc}$, and hence $\mathrm{d}V_g$ defined in Eq. (3.5) is an $L^\infty_{\rm loc}$ differential n-form.

With the preceding preparations, we now state our geometric quadratic theorem on vector bundles over a semi-Riemannian manifold, first with a C^∞ metric g.

Theorem 3.1 *Let M be a semi-Riemannian manifold with a C^∞ metric g. Let E and F be two real C^∞ vector bundles over M. Consider a family of E-sections $\{u_\varepsilon\} \subset L^2_{\mathrm{loc}}(M;E)$, a differential operator $\mathcal{T} \in \mathrm{Diff}^m(M;E,F)$ for some $m \in \mathbb{R}_+$ with the principal symbol $\sigma_m(\mathcal{T}) : T^*M \to \mathrm{Hom}(E;F^{\mathbb{C}})$, and a quadratic polynomial $Q : \Gamma(E) \to \mathbb{R}$. If the following conditions hold:*

(C1) *$u_\varepsilon \rightharpoonup u$ weakly in $L^2_{\mathrm{loc}}(M;E)$,*
(C2) *$\{\mathcal{T}u_\varepsilon\}$ is pre-compact in $H^{-m}_{\mathrm{loc}}(M;F)$,*
(C3) *$Q \circ s = 0$ for all $s \in \Lambda_{\mathcal{T}}$, where the* cone *of $\mathcal{T}$ is defined by*

$$\Lambda_{\mathcal{T}} := \big\{ s \in \Gamma(E) \,:\, \sigma_m(\mathcal{T})(\xi)(s) = 0 \text{ for some } \xi \in T^*M \setminus \{0\} \big\},$$

then, for any $\psi \in C^\infty_c(M)$,

$$\lim_{\varepsilon \to 0} \int_M (Q \circ u_\varepsilon)(x)\, \psi(x)\, \mathrm{d}V_g(x) = \int_M (Q \circ u)(x)\, \psi(x)\, \mathrm{d}V_g(x).$$

Before presenting the proof, we make several remarks on Theorem 3.1:

(i) Theorem 3.1 is formulated globally and intrinsically on the semi-Riemannian manifold M, since symbol σ_m, cone $\Lambda_{\mathcal{T}}$, and the Sobolev spaces $H^\bullet$ of sections are all defined without referring to local coordinates. In addition, σ_m is defined only by using the differentiable structure of M, without resort to the Riemannian or semi-Riemannian structure. Therefore, cone $\Lambda_{\mathcal{T}}$ in (C3) depends only on the *algebraic* properties of $\mathcal{T}$.

(ii) In Theorem 3.1, we denote the target space of symbol $\sigma_m(\mathcal{T})$ by $\mathrm{Hom}(E;F^{\mathbb{C}})$, which is understood as the vector bundle of $\mathbb{R}$-bundle homomorphisms from E to the *complexification* of F, *i.e.*, $F^{\mathbb{C}} := F \otimes \mathbb{C}$. It is also common to write it as
$$\sigma_m(\mathcal{T}) \in \Gamma(TM \otimes \mathrm{Hom}(E;F^{\mathbb{C}})).$$

(iii) The following lemma concerns the *naturality* of the principal symbol under the action of diffeomorphism group. It is crucial for the proof of Theorem 3.1.

Lemma 3.1 *Let $\mathcal{O}$ and $\tilde{\mathcal{O}}$ be open subsets of $\mathbb{R}^n$, and let $F : \mathcal{O} \to \tilde{\mathcal{O}}$ be a diffeomorphism. Then $F_*P \in \mathrm{Diff}^m(\tilde{\mathcal{O}})$ for $P \in \mathrm{Diff}^m(\mathcal{O})$. Moreover, the principal symbols of P and F_*P, i.e., $\sigma_m(F_*P)$ and $\sigma_m(P)$, are related as*

$$\sigma_m(F_*P)(F(x),\xi) = \sigma_m(P)(x, [\mathrm{d}_x F]^\top(\xi)) \quad \text{for each } x \in \mathcal{O} \text{ and } \xi \in T^*\mathbb{R}^n,$$

where F_*P denotes the pushforward *of P under F:*

$$(F_*P)(\varphi) = P(\varphi \circ F) \circ F^{-1} \qquad \text{for all } \varphi \in C^\infty_c(\tilde{\mathcal{O}}).$$

This is a special case of Theorem 20 in [1]. In full generality, the first assertion holds for general pseudo-differential operators, and the second assertion holds for pseudo-differential operators with classical total symbols.

The strategy for the proof of Theorem 3.1 is as follows: First of all, using a partition-of-unity, together with the commutator estimate of $\mathcal{T} \in$

$\mathrm{Diff}^m(M; E, F)$ and a multiplication operator, we reduce the theorem to a local problem on one single chart of the manifold. Next, thanks to Lemma 3.1, we can flatten the local chart to $\mathbb{R}^n$; this cannot be done directly, owing to the non-trivial semi-Riemannian metrics on the manifold and the bundles. Nevertheless, in view of the quadratic structure of Q, the *signature* of the semi-Riemannian metrics does not affect the proof. Therefore, locally we can regard the metrics as "close" to the Euclidean metrics, and then modify the arguments by Tartar [59] to complete the proof.

Proof of Theorem 3.1. The proof is divided into eight steps.

1. We first justify the following two reductions:

(i) It suffices to prove the theorem for $u = 0$. Indeed, we note that

$$Q(u_\varepsilon - u) = Q(u_\varepsilon) + Q(u) - 2\sum_{i,j=1}^{J} Q_{ij} u_\varepsilon^i u^j$$

for Q is a real quadratic polynomial. Condition (C1) yields

$$\sum_{i,j=1}^{J} Q_{ij} u_\varepsilon^i u^j \rightharpoonup Q(u) \qquad \text{weakly in } L^2_{\mathrm{loc}}.$$

Thus, $Q(u_\varepsilon - u)$ and $Q(u_\varepsilon) - Q(u)$ have the same distributional limit as $\varepsilon \to 0$.

(ii) We can *localize* the statement to each chart of the differentiable manifold M. To fix the notations, let $\{U^k\}_{k\in\mathcal{I}}$ be an atlas of the differentiable manifold M. We *claim* that it suffices to prove Theorem 3.1 for sequence $\{u_\varepsilon\}$ supported on one single U^k.

For this purpose, take any $\psi \in C_c^\infty(M)$ and consider the following identity:

$$\mathcal{T}(\psi u_\varepsilon) = \psi \mathcal{T} u_\varepsilon + [\mathcal{T}, \psi] u_\varepsilon,$$

where $[\mathcal{T}, \psi]$ denotes the commutator of $\mathcal{T}$ and the operator of multiplication by ψ.

Clearly, $[\mathcal{T}, \psi]$ is a differential operator of order not exceeding $m-1$. Since $\{u_\varepsilon\}$ is pre-compact (hence uniformly bounded) in L^2_{loc}, $\{[\mathcal{T}, \psi] u_\varepsilon\}$ is uniformly bounded in H^{-m+1}_{loc}, which is compactly embedded in H^{-m}_{loc} by the Rellich lemma. Moreover, by condition (C2), $\{\psi \mathcal{T} u_\varepsilon\}$ is also pre-compact in H^{-m}_{loc}. Thus, the same holds for $\{\mathcal{T}(\psi u_\varepsilon)\}$. In addition, the transition function $\varphi^{k,l}$ between any two overlapping charts U^k and U^l is a diffeomorphism, so that both $\mathcal{T}|_{U^k}$ and $\mathcal{T}|_{U^l}$ have the principal symbols of order m, which are m-homogeneous polynomials in the fiber of the cotangent bundle T^*M. Indeed, they differ only by a multiplicative factor controlled by the Lipschitz norm of $\varphi^{k,l}$, which is bounded uniformly on M for all $k, l \in \mathcal{I}$. Up to now, we have justified that the assumptions of the theorem are invariant under operation $u_\varepsilon \mapsto \psi u_\varepsilon$, where $\psi \in C_c^\infty(M)$ is an arbitrary test function.

It remains to establish the *local-to-global* result: If the assertion holds for $\{u_\varepsilon\}$ supported in each chart, then it also holds for arbitrary $\{u_\varepsilon\}$. To this end, let $\{\phi_k\}_{k\in\mathcal{I}}$ be a partition-of-unity subordinate to atlas $\{U^k\}_{k\in\mathcal{I}}$, *i.e.*, $0 \leq \phi^k \leq 1$, $\phi^k \in C_c^\infty(U^k)$ for each $k \in \mathcal{I}$, and $\sum_{k\in\mathcal{I}} \phi^k = 1$ on M. Then we can find $\psi^k \in C_c^\infty(U^k)$ with $0 \leq \psi^k \leq 1$ such that $\phi^k = (\psi^k)^2$ for each $k \in \mathcal{I}$. To proceed, suppose that Theorem 3.1 is proved for sequence $\{\psi^k u_\varepsilon\} \subset L^2(U^k; E)$ for each $k \in \mathcal{I}$, with $\psi^k u_\varepsilon \rightharpoonup \psi^k u$ in $L^2(U^k; E)$ along some subsequence $\{\psi^k u_{\varepsilon_i}\}_{i\in\mathcal{I}_1\subset\mathcal{I}}$. Then, for a neighboring chart U^l, *i.e.*, $U^k \cap U^l \neq \emptyset$, we can select a further subsequence $\{\psi^l u_{\varepsilon_j}\}_{j\in\mathcal{I}_2} \subset \{\psi^k u_{\varepsilon_i}\}_{i\in\mathcal{I}_1}$ such that $\mathcal{I}_2 \subset \mathcal{I}_1$, and $\{\psi^l u_{\varepsilon_j}\}$ converges weakly in $L^2(U^l; E)$ to some $\psi^l \tilde{u}$. However, due to the uniqueness of subsequential weak limits, we have

$$u = \tilde{u} \qquad \text{on } U^k \cap U^l.$$

Hence, we can write $\psi^l \tilde{u}$ as $\psi^l u$ without ambiguity, according to the interpretation: the limit function u, previously defined only on U^k, is now extended to domain $U^k \cup U^l$.

Now, since M is second-countable (which is a part of the definition of differentiable manifolds), we can take the index set $\mathcal{I}$ for the atlas to be at most countable. Thus, performing a diagonalization process to the arguments in the preceding paragraph, we obtain a subsequence (still denoted) $\{u_\varepsilon\}$ and a function $u \in L^2_{\mathrm{loc}}(M; E)$ defined on manifold M such that

$$\psi^k u_\varepsilon \rightharpoonup \psi^k u \qquad \text{for each } k \in \mathcal{I}.$$

Therefore, for any test function $\psi \in C_c^\infty(M)$, we can pass to the limit as follows:

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_M (Q \circ u_\varepsilon)(x)\psi(x)\,\mathrm{d}V_g(x) &= \sum_{k\in\mathcal{I}} \lim_{\varepsilon\to 0} \int_M (\psi^k)^2(x)\,(Q \circ u_\varepsilon)(x)\psi(x)\,\mathrm{d}V_g(x) \\
&= \sum_{k\in\mathcal{I}} \lim_{\varepsilon\to 0} \int_M Q(\psi^k(x)u_\varepsilon(x))\psi(x)\,\mathrm{d}V_g(x) \\
&= \sum_{k\in\mathcal{I}} \int_M Q(\psi^k(x)u(x))\psi(x)\,\mathrm{d}V_g(x) \\
&= \sum_{k\in\mathcal{I}} \int_M (\psi^k)^2(x)\,(Q \circ u)(x)\psi(x)\,\mathrm{d}V_g(x) \\
&= \int_M (Q \circ u)(x)\psi(x)\,\mathrm{d}V_g(x).
\end{aligned} \tag{3.6}$$

In the first and the last lines of (3.6), we have used that $\sum_{k\in\mathcal{I}}(\psi^k)^2 = 1$ on M, while in the second and the fourth lines, we have used the quadratic structure of Q. Moreover, the order of summation over $\alpha \in \mathcal{I}$ can be interchanged with the limit and the integration, because the partition-of-unity is locally finite. Then the localization argument is completed by using Eq. (3.6).

2. From now on, $\{u_\varepsilon\}$ is assumed to be supported on a single chart $U^k \subset M$. In this step, we *flatten the chart* by transforming U^k to $\mathbb{R}^n$ via the coordinate map. First, without loss of generality, we assume that the vector bundles E and F are trivialized on U^k; otherwise, a refinement of atlas $\{U^k\}_{k\in\mathcal{I}}$ can be made if necessary. Now, by the basic manifold theory, there exists a diffeomorphism $F^k : U^k \xrightarrow{\sim} \mathbb{R}^n$ so that

$$F^k_* \mathcal{T} \in \mathrm{Diff}^m(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^J, \mathbb{R}^n \times \mathbb{R}^I).$$

Here and hereafter, we assume that E and F have typical fibers $\mathbb{R}^J$ and $\mathbb{R}^I$, respectively.

Moreover, Lemma 3.1 implies

$$\sigma_m(F^k_*\mathcal{T})(F^k(x), \zeta) = \sigma_m(\mathcal{T})(x, [\mathrm{d}_x F^k]^\top(\zeta)) \quad \text{for all } x \in U^k \text{ and } \zeta \in \mathbb{R}^J. \tag{3.7}$$

Notice that ζ and $[\mathrm{d}_x F^k]^\top(\zeta)$ are simultaneously non-vanishing in Eq. (3.7), since F^k is a diffeomorphism. We conclude

$$\Lambda_{\mathcal{T}} = \Lambda_{F^k_*\mathcal{T}},$$

i.e., the cones of $\mathcal{T}$ and $F^k_*\mathcal{T}$ coincide.

Therefore, it suffices to prove the theorem with $\{\mathrm{d}F^k(\psi^k u_\varepsilon)\}$ and $F^k_*\mathcal{T}$ in place of $\{u_\varepsilon\}$ and $\mathcal{T}$, respectively, where $\{\psi^k : k \in \mathcal{I}\}$ is a partition-of-unity subordinate to atlas $\{U^k : k \in \mathcal{I}\}$ as in Step 1. In addition, by the paracompactness of topological manifolds, we may assume ψ^k to be supported in a compact subset of U^k for each $k \in \mathcal{I}$. Thus, in the sequel, we take $\mathrm{d}F^k(\psi^k u_\varepsilon)$ to be compactly supported in $\mathbb{R}^n$ and identify it with the map on the whole of $\mathbb{R}^n$, obtained via the extension-by-zero. To simplify the notations, we still label $\{\mathrm{d}F^k(\psi^k u_\varepsilon)\}$ as $\{u_\varepsilon\}$. Thus, we reduce to the case: $M = \mathbb{R}^n$.

3. Thanks to the localization and flattening arguments in Steps 1–2, from now on, we assume $\{u_\varepsilon\} \subset L^2_c(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^J)$ and $\mathcal{T} \in \mathrm{Diff}^m(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^J; \mathbb{R}^n \times \mathbb{R}^I)$. To simplify the notations, we still write $E = \mathbb{R}^n \times \mathbb{R}^J$ and $F = \mathbb{R}^n \times \mathbb{R}^I$, and denote the metric on $\mathbb{R}^n$ by g with an abuse of notations, *i.e.*, assuming that $M = \mathbb{R}^n$, and the bundles E and F are globally trivialized.

To begin with, recall that the L^p norm of $u : \mathbb{R}^n \to E$ is defined as

$$\begin{aligned} \|u\|_{L^p(\mathbb{R}^n;E)} &:= \Big(\int_{\mathbb{R}^n} |u|^2_{g^E} \, \mathrm{d}V_g \Big)^{\frac{1}{p}} \\ &= \Big(\int_{\mathbb{R}^n} \Big\{ \sum_{j=1}^J \sum_{k=1}^J \epsilon^k g^E_{jk}(x) u^j(x) u^k(x) \Big\}^{\frac{p}{2}} \sqrt{|\det\, g(x)|} \, \mathrm{d}x \Big)^{\frac{1}{p}}, \end{aligned}$$

where g^E is the bundle metric on E, indices $1 \le j, k \le J$ are for the fiber of E, and $\epsilon^k \in \{\pm 1\}$ is the *signature* of the k-th component of g^E such that $(h_{jk}) := (\epsilon^k g^E_{jk})$ becomes positive definite. Here and in the sequel, we choose a coordinate system in which g^E is diagonalized:

$$g^E = \mathrm{diag}(\lambda^1, \dots, \lambda^\tau; \lambda^{\tau+1}, \dots, \lambda^J),$$

where $\lambda^j < 0$ for $1 \leq j \leq \tau$, and $\lambda^j > 0$ for $\tau+1 \leq j \leq J$. Correspondingly, $\epsilon^1 = \cdots = \epsilon^\tau = -1$ and $\epsilon^{\tau+1} = \cdots = \epsilon^J = 1$, where τ is the index of g.

Now, define a new sequence of sections $\{v_\varepsilon\} \subset L^2(\mathbb{R}^n; E)$ by components:

$$v_\varepsilon^j := \sqrt{\epsilon^j \lambda^j}\,|\det g|^{\frac{1}{4}} u_\varepsilon^j \qquad \text{for each } j = 1, 2, \ldots, J. \tag{3.8}$$

That is, we write $v_\varepsilon = (v_\varepsilon^1, \ldots, v_\varepsilon^J)^\top$. By this definition, v_ε depends on g^E, g, and u_ε, and the following identity holds:

$$\|v_\varepsilon\|^2_{L^2(\mathbb{R}^n,\, g_0; E)} \equiv \sum_{j=1}^J \int_{\mathbb{R}^n} \{v_\varepsilon^j(x)\}^2 \,\mathrm{d}x = \|u_\varepsilon\|^2_{L^2(\mathbb{R}^n,\, g; E)} \quad \text{for each } \varepsilon, \tag{3.9}$$

where g_0 denotes the *Euclidean* metric on $\mathbb{R}^n$. Thus, by condition (C1), $\{v_\varepsilon\}$ is uniformly bounded in L^2 with respect to g_0. Moreover, $\mathrm{supp}(v_\varepsilon) \subset \mathrm{supp}(u_\varepsilon)$ for each ε so that all the terms of $\{v_\varepsilon\}$ are supported on a common compact set. By the Riemann-Lebesgue lemma, there are finite numbers $K, K' > 0$ such that $\|\hat{v}_\epsilon\|_{L^\infty(B_K)} \leq K'$, where B_K is the Euclidean ball $\{\xi \in \mathbb{R}^n : |\xi| < K\}$. Thanks to the Parseval identity, the Cauchy–Schwarz inequality, and (C1), we now have

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{|\xi|\leq K} |\hat{v}_\varepsilon(\xi)|^2 \,\mathrm{d}\xi &\leq K' \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \hat{v}_\varepsilon \overline{\mathrm{sgn}(\hat{v}_\varepsilon)} \chi_{B_K} \,\mathrm{d}\xi \\ &\leq K'\sqrt{K^n} \lim_{\varepsilon\to 0} \|\hat{v}_\varepsilon\|_{L^2(\mathbb{R}^n, g_0; E)} = 0, \end{aligned} \tag{3.10}$$

where $\mathrm{sgn}(z) := \frac{z}{|z|}$ for $z \neq 0$, the choice of K' is immaterial, and K will be further specified later.

As a remark, the norm on ξ is also taken with respect to the Euclidean metric, since it is the metric induced by g_0 on the cotangent bundle T^*M.

4. Next we control the high-frequency region of $\{v_\varepsilon\}$. For $j = 1, 2, \ldots, J$, define

$$\chi^j = \chi^j(g) := |\det g|^{\frac{1}{4}} \sqrt{\epsilon^j \lambda^j},$$

so that $v_\varepsilon^j = \chi^j u_\varepsilon^j$ for each j. Notice that $\chi^j > 0$ strictly, by the non-degeneracy of metrics g and g^E. Writing $u_\varepsilon = \sum_{j=1}^J u_\varepsilon^j \partial_j$ and similarly for v_ε in local coordinates, by the linearity of the differential operator $\mathcal{T}$, we have

$$\begin{aligned}\mathcal{T} v_\varepsilon &= \mathcal{T}\Big\{\sum_{j=1}^J \chi^j u_\varepsilon^j \partial_j\Big\} \\ &= \sum_{j=1}^J \chi^j\, \mathcal{T}(u_\varepsilon^j \partial_j) + \sum_{j=1}^J [\mathcal{T}, \chi^j] u_\varepsilon^j \partial_j =: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon.\end{aligned}$$

In II_ε, $[\mathcal{T}, \chi^j]$ is the commutator between $\mathcal{T}$ and the multiplication operator by χ^j.

We now argue that $\{\mathcal{T} v_\varepsilon\}$ *is pre-compact in* $H^{-m}(\mathbb{R}^n, g_0; F)$. First of all, this sequence is compactly supported, by the construction of $\{v_\varepsilon\}$ and the locality of the differential operator $\mathcal{T}$. Thus, we neglect subscript "loc" for the corresponding Sobolev spaces. By explicitly writing out g_0 in the subscript, we emphasize that $M = \mathbb{R}^n$ is equipped with the Euclidean metric. To this end, we now prove that both $\{\mathrm{I}_\varepsilon\}$ and $\{\mathrm{II}_\varepsilon\}$ are pre-compact in $H^{-m}(\mathbb{R}^n, g_0; F)$.

For I_ε, we first compute:

$$\begin{aligned}\|\mathrm{I}_\varepsilon\|_{H^{-m}(\mathbb{R}^n,g_0;F)} &\leq \sup_{1\leq j\leq J} \|\chi^j - 1\|_{L^\infty(\mathbb{R}^n)} \|Tu_\varepsilon\|_{H^{-m}(\mathbb{R}^n,g_0;F)} \\ &\leq \big(1 + \|g^E\|^{\frac{1}{2}}_{L^\infty(E)} \|\det g\|^{\frac{1}{4}}_{L^\infty(M)}\big) \|Tu_\varepsilon\|_{H^{-m}(\mathbb{R}^n,g_0;F)}.\end{aligned} \tag{3.11}$$

Next, we show that the final term $\|Tu_\varepsilon\|_{H^{-m}(\mathbb{R}^n,g_0;F)}$ can be related to $\|Tu_\varepsilon\|_{H^{-m}(\mathbb{R}^n,g;F)}$, whose pre-compactness is assumed by condition (C2). For this purpose, it requires to invoke the Fourier characterization of the Sobolev norms $\|\cdot\|_{H^{-m}(\mathbb{R}^n,g_0;F)}$ and $\|\cdot\|_{H^{-m}(\mathbb{R}^n,g;F)}$. Since we have localized sequence $\{u_\varepsilon\}$ to a chart U^k of M, on which E and F are trivialized in Steps 1–2, $g|_{U^k}$ has no self-intersecting geodesics, provided that U^k is contained in a geodesic normal neighborhood. This can be assumed by shrinking U^k if necessary. Then the pushforward metric $F^k_* g$ — which is still labelled as g from Step 2 onward — satisfies the same property on $\mathbb{R}^n = M$, so that $\|\cdot\|_{H^{-m}(\mathbb{R}^n,g;F)}$ can be defined globally via the Fourier transform unambiguously.

In this way, we now obtain

$$\begin{aligned}\|\mathcal{T}u_\varepsilon\|_{H^{-m}(\mathbb{R}^n,g_0;F)} &= \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{T}u_\varepsilon}(\xi)|^2_{g^F}}{(1+|\xi|^2)^{m/2}}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{T}u_\varepsilon}(\xi)|^2_{g^F}}{(1+|\xi|^2_g)^{m/2}} \sqrt{|\det g|} \left\{\frac{(1+|\xi|^2_g)^{m/2}}{(1+|\xi|^2)^{m/2}} \frac{1}{\sqrt{|\det g|}}\right\} \mathrm{d}\xi \\ &\leq C \int_{\mathbb{R}^n} \frac{|\widehat{\mathcal{T}u_\varepsilon}(\xi)|^2_{g^F}}{(1+|\xi|^2_g)^{m/2}} \sqrt{|\det g|}\,\mathrm{d}\xi \\ &=: C\|\mathcal{T}u_\varepsilon\|_{H^{-m}(\mathbb{R}^n,g;F)},\end{aligned} \tag{3.12}$$

where C depends only on m, $\|g\|_{L^\infty}(M)$, and $\inf_M |\det g|$. Together with Eq. (3.11), we have

$$\|\mathrm{I}_\varepsilon\|_{H^{-m}(\mathbb{R}^n,g_0;F)} \leq \tilde{C}\|\mathcal{T}u_\varepsilon\|_{H^{-m}(\mathbb{R}^n,g;F)},$$

where $\tilde{C}$ depends only on g, g^E, and m, but independent of ε. In view of (C2), $\{\mathrm{I}_\varepsilon\}$ is pre-compact in $H^{-m}(\mathbb{R}^n, g_0; F)$.

We now turn to $\{\mathrm{II}_\varepsilon\}$: Since $\mathcal{T} \in \mathrm{Diff}^m(M; E, F)$ and χ^j is a multiplication operator, $[\mathcal{T}, \chi^j] \in \mathrm{Diff}^r(M; E, F)$ for $r \leq m-1$. By assumption (C1), $\{u_\varepsilon\}$ is bounded in $L^2(M, g; E)$, hence $\{\mathrm{II}_\varepsilon\}$ is pre-compact in $H^{-m}(\mathbb{R}^n, g; F)$ due to the Rellich lemma. Again, by the estimates in Eq. (3.12), $\{\mathrm{II}_\varepsilon\}$ is also pre-compact in $H^{-m}(\mathbb{R}^n, g_0; F)$.

Therefore, $\{\mathcal{T}v_\varepsilon\}$ is pre-compact in $H^{-m}(\mathbb{R}^n, g_0; F)$ so that

$$\mathcal{T}v_\varepsilon \longrightarrow 0 \qquad \text{strongly in } H^{-m}(\mathbb{R}^n, g_0; F). \tag{3.13}$$

This is because $u_\varepsilon \rightharpoonup 0$ in L^2 (see Step 1 above). Here $\mathbb{R}^n$ is endowed with the Euclidean metric g_0, and F has the bundle metric g^F.

5. Now we estimate the Euclidean L^2 norm of $\widehat{\mathcal{T}v_\varepsilon}$ on $\{|\xi|\geq 1\}$, where $\widehat{\mathcal{T}v_\varepsilon}$ is the standard Fourier transform on Euclidean spaces:

$$\widehat{\mathcal{T}v_\varepsilon}(\xi) := \int_{\mathbb{R}^n} \mathcal{T}v_\varepsilon(x) e^{-2\pi i \xi\cdot x}\,\mathrm{d}x.$$

Indeed, since $\mathcal{T}\in \mathrm{Diff}^m(M;E,F)$, by the localization and flattening in Steps 1–2, we have

$$\mathcal{T} = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha,$$

and the principal symbol is given by

$$\sigma_m(\mathcal{T})(x,\xi) = (-2\pi i)^m \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha;$$

see §3 in [1]. Combining with the lower order terms, we have

$$\widehat{\mathcal{T}v_\varepsilon}(\xi) = (-2\pi i)^m \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \hat{v}_\varepsilon(\xi) + \sum_{|\beta|\leq m-1} b_\beta(x,\xi)\xi^\beta \hat{v}_\varepsilon(\xi),$$

where $|a_\alpha(x)|+|b_\beta(x,\xi)|\leq C_0$ for all $x\in M$ and $\xi\in T_x^*M$, and for each α and β. Then

$$\begin{aligned}
&\|\mathcal{T}v_\varepsilon\|^2_{H^{-m}(\{|\xi|\geq K\})}\\
&:= \int_{|\xi|\geq K} \frac{\big|(-2\pi i)^m \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \hat{v}_\varepsilon(\xi) + \sum_{|\beta|\leq m-1} b_\beta(x,\xi)\xi^\beta \hat{v}_\varepsilon(\xi)\big|^2}{(1+|\xi|^2)^m}\,\mathrm{d}\xi\\
&\geq C_1^{-1} \sum_{|\alpha|=m} \int_{|\xi|\geq K} \frac{|a_\alpha(x)|^2|\xi|^{2m}}{(1+|\xi|^2)^m} |\hat{v}_\varepsilon(\xi)|^2\,\mathrm{d}\xi\\
&\qquad - C_2 \sum_{|\gamma|\leq 2m-1} \int_{|\xi|\geq K} \frac{|\xi|^\gamma}{(1+|\xi|^2)^m} |\hat{v}_\varepsilon(\xi)|^2\,\mathrm{d}\xi,
\end{aligned}$$

where C_1 depends only on m, while $C_2 = C_2(\sup_x |b_\beta(x)|, m)$. This is obtained by expanding the quadratic in the second line above and separating the highest order term from the other terms. Now, choosing $K\geq 1$ so large that the second term is majorized by the first term in the last line, we have

$$\|\mathcal{T}v_\varepsilon\|^2_{H^{-m}(\{|\xi|\geq K\})} \geq C_3^{-1} \sum_{|\alpha|=m} \int_{|\xi|\geq K} \frac{|a_\alpha(x)|^2|\xi|^{2m}}{(1+|\xi|^2)^m} |\hat{v}_\varepsilon(\xi)|^2\,\mathrm{d}\xi,$$

which converges to 0 by Eq. (3.13), where C_3 depends on C_1, C_2, and K.

6. In this step, we complexify Q and $\Lambda_{\mathcal{T}}$. First, we view $Q : \Gamma(E) \to \mathbb{R}$ as a complex quadratic polynomial $Q^{\mathbb{C}} : \Gamma(E^{\mathbb{C}}) \to \mathbb{C}$, given by the following expression in local coordinates:

$$Q^{\mathbb{C}}(z) := Q^{\mathbb{C}}_x(z) := \sum_{j,k} Q_{jk}(x) z^j \overline{z^k} \qquad \text{for } x \in M \text{ and } z \in E^{\mathbb{C}}_x,$$

where $E^{\mathbb{C}}_x \cong \mathbb{C}^J$ is the fiber of the complexified bundle $E^{\mathbb{C}} := E \otimes \mathbb{C}$ at point x. Thus, $Q^{\mathbb{C}}(s) = Q(s)$ for real $s \in \Gamma(E)$. Moreover, we define the *complexified cone* by

$$\Lambda^{\mathbb{C}}_{\mathcal{T}} := \Lambda_{\mathcal{T}} + i\Lambda_{\mathcal{T}} = \{s^{\mathbb{C}} = s + is \,:\, s \in \Lambda_{\mathcal{T}}\}.$$

We now compute $Q^{\mathbb{C}}(\zeta)$ for $\zeta = s + ir$, where $s, r \in \Gamma(E)$ are real: Indeed,

$$Q^{\mathbb{C}}(s+ir) = Q(s) + Q(r) + i\big\{\mathbf{q}(r,s) + \mathbf{q}(s,r)\big\},$$

where $\mathbf{q}(r,s) := \sum_{j,k} Q_{jk}(x) r^j \overline{s^k}$ and $Q(s) = \mathbf{q}(s,s)$ as before. In particular, we have

$$Q^{\mathbb{C}}(s^{\mathbb{C}}) = 2Q(s) + 2iQ(s) = 2Q(s)^{\mathbb{C}},$$

so that, for $s^{\mathbb{C}} = s + is \in \Lambda^{\mathbb{C}}_{\mathcal{T}}$, the following facts hold:

(i) $Q(s) >, =, \text{or} < 0$ if and only if $\mathrm{Re}\{Q^{\mathbb{C}}(s^{\mathbb{C}})\} >, =, \text{or} < 0$ (respectively);

(ii) $s \in \Lambda_{\mathcal{T}}$ if and only if $s^{\mathbb{C}} \in \Lambda^{\mathbb{C}}_{\mathcal{T}}$;

(iii) For any $\psi \in C^\infty_c(M)$ and $(u_\varepsilon)^{\mathbb{C}} := u_\varepsilon + iu_\varepsilon$, we have

$$\lim_{\varepsilon\to 0} \int_M (Q \circ u_\varepsilon)\psi \,\mathrm{d}V_g = 0 \qquad \Longleftrightarrow \qquad \lim_{\varepsilon\to 0} \int_M \mathrm{Re}\{Q^{\mathbb{C}} \circ (u_\varepsilon)^{\mathbb{C}}\}\psi \,\mathrm{d}V_g = 0.$$

7. We first observe the following pointwise inequality: For each $\delta > 0$ and any compact set $\mathcal{K} \Subset T^*M \setminus \{0\}$, there is a constant $C_{\delta,\mathcal{K}} \in (0,\infty)$ such that

$$\mathrm{Re}\{Q^{\mathbb{C}}(s^{\mathbb{C}})\} \geq -\delta \big||s^{\mathbb{C}}|_{g^{E,\mathbb{C}}}\big|^2 - C_{\delta,\mathcal{K}} \big||\sigma_m(T)(\eta)(s^{\mathbb{C}})|_{g^{F,\mathbb{C}}}\big|^2 \tag{3.14}$$

for each $\eta \in \mathcal{K}$ and $s \in \Gamma(E)$, provided that $\mathrm{Re}(Q) \geq 0$ on $\Lambda_{\mathcal{T}}$. Here $g^{E,\mathbb{C}}$ is the *complexified bundle metric on* E, obtained according to the same rule for $Q \mapsto Q^{\mathbb{C}}$, by viewing g^E as a quadratic form on each fiber (*i.e.*, a vector space) of E; and similarly for $g^{F,\mathbb{C}}$.

Indeed, since Eq. (3.14) is 2-homogeneous in $s^{\mathbb{C}}$, the scaling: $s^{\mathbb{C}} \mapsto \lambda s^{\mathbb{C}}$ by any $\lambda \in \mathbb{C}$ leaves it invariant. In particular, it is independent of the signatures of the semi-Riemannian bundle metrics g^E and g^F. Moreover, cone $\Lambda_{\mathcal{T}}$ in (C3) is completely determined by $\mathcal{T}$, which is independent of metrics g, g^E, and g^F, and sequences $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$. Thus, Eq. (3.14) follows from a simple contradictory argument as in Tartar's proof of the classical quadratic theorem [59].

We now integrate Eq. (3.14) over $\{|\xi| \geq K\}$, with $K \geq 1$ specified at the end of Step 5 above, $s^{\mathbb{C}} = (\hat{v}_\varepsilon)^{\mathbb{C}}$, and $\eta := \frac{\xi^{2m}}{(1+|\xi|^2)^m}$. Then

$$2^{-m} = \frac{|\xi|^{2m}}{(2|\xi|^2)^m} \leq |\eta| \leq 1 \qquad \text{for all } |\xi| \geq K.$$

We remark here that it is crucial for sequence $\{v_\varepsilon\}$ to be taken on $M=\mathbb{R}^n$ with respect to the Euclidean metric (*cf.* Step 3 above). In this case, the metric induced on the cotangent bundle T^*M is also Euclidean, so that $|\xi|\neq 0$ for all $\xi\in T^*M\setminus\{0\}$.

To proceed, $\eta\in\mathcal{K}:=\{2^{-m}\leq|\xi|\leq 1\}$ is indeed a compact subset of $T^*M\setminus\{0\}$ so that

$$\begin{aligned}&\int_{|\xi|\geq K}\mathrm{Re}\{Q^{\mathbb{C}}(\hat{v}_\varepsilon)^{\mathbb{C}}\}\,\mathrm{d}\xi\\&\geq -\delta\|v_\varepsilon\|^2_{L^2(\mathbb{R}^N;E)}-C_{\delta,\mathcal{K}}\Big\{\sum_{|\alpha|=m}\int_{|\xi|\geq K}\frac{|a_\alpha(x)|^2|\xi|^{2m}}{(1+|\xi|^2)^m}|\hat{v}_\varepsilon(\xi)|^2\,\mathrm{d}\xi\Big\},\end{aligned}$$

where the last term on the right-hand side tends to zero as $\varepsilon\to 0$ (*cf.* Step 5). Therefore, we have

$$\lim_{\varepsilon\to 0}\int_{|\xi|\geq K}\mathrm{Re}\{Q^{\mathbb{C}}(\hat{v}_\varepsilon)^{\mathbb{C}}\}\,\mathrm{d}\xi\geq -C_0\delta\qquad\text{for arbitrary }\delta>0,$$

where $C_0=\sup_{\varepsilon>0}\|v_\varepsilon\|^2_{L^2(\mathbb{R}^n;E)}=\sup_{\varepsilon>0}\|u_\varepsilon\|^2_{L^2(\mathbb{R}^n,\,g;E)}<\infty$. This implies that the left-hand side is non-negative. Applying the same argument for $-Q$ in place of Q, thanks to condition (C3) and Step 6 above, we finally obtain

$$\lim_{\varepsilon\to 0}\int_{|\xi|\geq K}\mathrm{Re}\,\{Q^{\mathbb{C}}\circ\hat{v}_\varepsilon{}^{\mathbb{C}}\}(\xi)\,\mathrm{d}\xi=0,$$

that is,

$$\lim_{\varepsilon\to 0}\int_{|\xi|\geq K}\mathrm{Re}\,\{Q(\hat{v}_\varepsilon)\}(\xi)\,\mathrm{d}\xi=0.\tag{3.15}$$

8. Now we combine (3.10) with (3.15) and employ the Plancherel formula to conclude

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big|\int_{\mathbb{R}^n}\mathrm{Re}\{Q(\hat{v}_\varepsilon(\xi))\}\,\mathrm{d}\xi\Big|\\&\leq\lim_{\varepsilon\to 0}\int_{|\xi|<K}\big|\mathrm{Re}\{Q(\hat{v}_\varepsilon(\xi))\}\big|\,\mathrm{d}\xi+\lim_{\varepsilon\to 0}\Big|\int_{|\xi|\geq K}\mathrm{Re}\{Q(\hat{v}_\varepsilon(\xi))\}\,\mathrm{d}\xi\Big|\\&\leq C\lim_{\varepsilon\to 0}\int_{|\xi|<K}|\hat{v}_\varepsilon(\xi)|^2\,\mathrm{d}\xi=0,\end{aligned}\tag{3.16}$$

for some constant $C>0$ independent of $\varepsilon>0$. Then we infer from the Plancherel formula that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}(Q\circ v_\varepsilon)(x)\,\mathrm{d}x=0.$$

Also, recall from Equation (3.8) that v_ε differs from u_ε by a multiplicative factor depending only on the L^∞ norms of metrics on M and E (independent of ε). As Q is quadratic, we thus deduce

$$\lim_{\varepsilon\to 0}\int_M(Q\circ u_\epsilon)(x)\,\mathrm{d}V_g=0.$$

Moreover, we recall from Step 1 that the assertion of Theorem 3.1 is invariant under localizations, *i.e.*, multiplication by test functions $\psi \in C_c^\infty(M)$. Therefore, we can now conclude that $\{Q \circ u_\varepsilon\}$ converges to $Q \circ u$ in the sense of distributions. This completes the proof.

We emphasize that the non-degeneracy condition of metric, $\det g \neq 0$, is crucial to the proof. We need it in Eq. (3.12) to compare the H^{-m} norms of $\mathcal{T}\hat{u}^\varepsilon$ taken with respect to g and the Euclidean metric g_0. Therefore, we can extend Theorem 3.1 to a more general theorem, Theorem 3.2 below, for non-smooth metrics g, g^E, and g^F, which is crucial to the development in §4. Notice that, in the proof of Theorem 3.1, only the L^∞_{loc} topology of the metrics are involved in the estimates. Thus, in view of the Morrey–Sobolev embedding, the following result holds by an approximation argument:

Theorem 3.2 *Let M be a semi-Riemannian manifold with a non-degenerate L^∞_{loc} metric g (i.e., $|\det g| \geq \eta_0 > 0$ a.e.). Let E and F be two real vector bundles over M with L^∞_{loc} bundle metrics g^E and g^F, respectively. Consider a sequence of E-sections $\{u_\varepsilon\} \subset L^2(M;E)$, a differential operator $\mathcal{T} \in \mathrm{Diff}^m(M;E,F)$ for some $m \in \mathbb{R}_+$ with the principal symbol $\sigma_m(\mathcal{T}) : T^*M \to \mathrm{Hom}(E;F^{\mathbb{C}})$, and a real quadratic polynomial $Q : \Gamma(E) \to \mathbb{R}$. If the following conditions hold*:

(C-1) $u_\varepsilon \rightharpoonup u$ *weakly in* $L^2_{\mathrm{loc}}(M;E)$,

(C-2) $\{\mathcal{T}u_\varepsilon\}$ *is pre-compact in* $H^{-m}_{\mathrm{loc}}(M;F)$,

(C-3) $Q(s) = 0$ *for all* $s \in \Lambda_{\mathcal{T}}$, *where the* cone *of* $\mathcal{T}$ *is defined by*

$$\Lambda_{\mathcal{T}} := \big\{ s \in \Gamma(E) \,:\, \sigma_m(\mathcal{T})(\xi)(s) = 0 \text{ for some } \xi \in T^*M \setminus \{0\} \big\},$$

then

$$\lim_{\varepsilon \to 0} \int_M (Q \circ u_\varepsilon)\psi \,\mathrm{d}V_g = \int_M (Q \circ u)\psi \,\mathrm{d}V_g \qquad \text{for any } \psi \in C_c^\infty(M).$$

To conclude this section, besides the geometric theorem, Theorem 3.2, we can also obtain a generalized compensated compactness theorem in the abstract harmonic analysis settings. Although this result is not needed for our weak continuity theorem (Theorem 4.1) for the Cartan structural system below, it is of independent interest from the perspectives of compensated compactness and harmonic analysis. In addition, it may help to elucidate certain steps in the lengthy proof of Theorem 3.1 that leads to Theorem 3.2 above.

We first recall some basics of abstract harmonic analysis (*cf.* Loomis [42] and the notes by Tao [60]). A *topological group G* is a group with a topology, in which the group operation and the inverse are continuous. If a group G is Abelian whose topology is Hausdorff and locally compact, we say that G is a *locally compact Abelian group*, abbreviated as *LCA group* in the sequel. For any LCA group G, there exists an invariant Radon measure μ_G, unique up to multiplicative constants, known as the *Haar measure*. The L^p norm, $1 \leq p < \infty$, for a function $u : G \to \mathbb{C}$ can then be defined as

$$\|u\|_{L^p(G)} := \Big(\int_G |u(g)|^p \,\mathrm{d}\mu_G(g) \Big)^{1/p}.$$

Given any LCA group G, its *group of characters*, $\hat{G} := \mathrm{Hom}(G; \mathbb{R}/\mathbb{Z})$, is also an LCA group endowed with the local-uniform topology of any non-trivial Haar measure (which is the weakest topology making each element of $\hat{G}$ continuous). It is also known as the *dual* of G, due to the *Pontryagin duality theorem*: G is canonically isomorphic to $\hat{\hat{G}}$. Then, for $u \in L^1(G)$, we can define its *Fourier transform* $\hat{u} : \hat{G} \to \mathbb{C}$ by

$$\hat{u}(\xi) := \int_G u(g) e^{-2\pi i \xi(g)} \,\mathrm{d}\mu_G(g), \tag{3.17}$$

where $\xi(g)$ is given by the duality pairing of $\hat{G}$ and G. From now on, we write $0 \in \hat{G}$ as the group identity; this is in agreement with the definition, $\hat{G} := \mathrm{Hom}(G; \mathbb{R}/\mathbb{Z})$, which is the group of *additive* (not multiplicative) characters.

Next, the *Plancherel formula* extends to the general LCA groups:

$$\|u\|_{L^2(G)} = \|\hat{u}\|_{L^2(\hat{G})} \qquad \text{for all } u \in L^2(G),$$

with the Haar measures μ_G and $\mu_{\hat{G}}$ suitably normalized. In other words, the Fourier transform defined in Eq. (3.17) is an isometry between $L^2(G)$ and $L^2(\hat{G})$. Notice that all the constructions up to now can naturally be extended to vector-valued functions $u : G \to \mathbb{C}^I$ for $I \geq 1$.

Finally, we say that $\mathcal{T} : L^2(G) \to L^2(G)$ is a *multiplier operator* if

$$\widehat{\mathcal{T}u}(\xi) = m(\xi)\hat{u}(\xi) \qquad \text{for some } m : \hat{G} \to \mathbb{C},$$

where m is known as the *Fourier multiplier* of $\mathcal{T}$. More generally, for $\mathcal{T} : L^2(G; \mathbb{C}^J) \to L^2(G; \mathbb{C}^I)$ for $I, J \geq 1$, the multiplier is a mapping

$$m : \hat{G} \to \mathrm{Mat}(I \times J; \mathbb{C}) \cong (\mathbb{C}^J)^* \otimes \mathbb{C}^I.$$

That is, for each $\xi \in \hat{G}$, $m(\xi)$ is a linear operator from $\mathbb{C}^J$ to $\mathbb{C}^I$ (equivalently, an $I \times J$ matrix). In the sequel, for any matrix $M \in \mathrm{Mat}(I \times J; \mathbb{C})$, we use $|M| := \sqrt{\sum_{i=1}^I \sum_{j=1}^J |M_{ij}|^2}$ to denote its Hilbert–Schmidt norm.

In this context, we say that $Q : \mathbb{C}^N \to \mathbb{C}$ is a *quadratic polynomial* if it is a Hermitian 2-form on $\mathbb{C}^N$, *i.e.*, $Q = \{Q_{jk}\}$ as a complex $N \times N$ matrix satisfies

$$Q_{jk} = \overline{Q_{kj}} \qquad \text{for each } j, k = 1, 2, \ldots, N.$$

That is,

$$Q(\lambda) = \sum_{j,k=1}^N Q_{jk} \lambda^j \overline{\lambda^k} \quad \text{for } \lambda = (\lambda^1, \ldots, \lambda^N) \in \mathbb{C}^N \text{ and constants } Q_{jk} \in \mathbb{C}. \tag{3.18}$$

Theorem 3.3 *Let G be an LCA group with Haar measure μ_G. Consider a sequence $\{u_\varepsilon\}$ in $L^2_c(G;\mathbb{C}^J)$, a Fourier multiplier operator $\mathcal{T} : L^2(G;\mathbb{C}^J) \to H^{-s}(G;\mathbb{C}^I)$ with multiplier $m : \hat{G} \to \mathrm{Mat}(I \times J;\mathbb{C})$ for some $s \in \mathbb{R}_+$, and a quadratic polynomial $Q : \mathbb{C}^J \to \mathbb{C}$. Assume that*

(i) *$u_\varepsilon \rightharpoonup u$ weakly in $L^2(G;\mathbb{C}^J)$.*

(ii) *The end of $\hat{G}$ retracts nicely onto a compact set. More precisely, for some compact set $\Xi \Subset \hat{G}$ containing 0, there exist another compact set $\mathcal{K} \Subset \hat{G}\backslash\{0\}$ and a continuous surjective map $\Phi : \hat{G}\backslash\Xi \to \mathcal{K}$ such that $\{(\Phi^* m)\hat{u_\varepsilon}\}$ is pre-compact in $L^2(\hat{G} \setminus \Xi;\mathbb{C}^J)$.*

(iii) *$Q(\lambda) = 0$ for all $\lambda \in \Lambda_{\mathcal{T}}$, where $\Lambda_{\mathcal{T}}$ (the* cone *of $\mathcal{T}$) is defined by*

$$\Lambda_{\mathcal{T}} := \big\{\lambda \in \mathbb{C}^J \,:\, m(\xi)(\lambda) = 0 \;\text{ for some } \xi \in \hat{G} \setminus \{0\}\big\}. \tag{3.19}$$

Then

$$\lim_{\varepsilon\to 0} \int_G (Q \circ u_\varepsilon)(g)\, \mathrm{d}\mu_G(g) = \int_G (Q \circ u)(g)\, \mathrm{d}\mu_G(g).$$

In Theorem 3.3 above, the *pullback* of m under Φ, *i.e.*, $\Phi^* m : \hat{G} \setminus \Xi \to [0,\infty]$, is given by $\Phi^* m(\xi) := m(\Phi(\xi))$. In the definition of $\Lambda_{\mathcal{T}}$ in (3.19), we view $m : \hat{G} \to (\mathbb{C}^J)^* \otimes \mathbb{C}^I$. That is, $m(\xi)$ is an operator from $\mathbb{C}^J$ to $\mathbb{C}^I$ so that $m(\xi)(\lambda) \in \mathbb{C}^I$. According to this interpretation, another characterization of the cone is

$$\Lambda_{\mathcal{T}} = \bigcup_{\xi \in \hat{G}\backslash\{0\}} \ker[m(\xi)].$$

The proof of Theorem 3.3 can be found in Appendix B.

4 Global Weak Continuity of the Cartan Structural System

In this section, we establish the weak continuity of the Cartan structural system (2.13) on semi-Riemannian manifolds. The arguments are global and intrinsic, based on the geometric compensated compactness theorem, Theorem 3.2. This extends our earlier results on the weak continuity of the GCR system on Riemannian manifolds [12,14].

Theorem 4.1 *Let (M,g) be a semi-Riemannian manifold of dimension n, with $\mathrm{Ind}(M) = \nu$, $g \in L^\infty_{\mathrm{loc}}$, and the Levi–Civita connection ∇ of g in L^p_{loc} for $p > 2$. Assume that a family of connection 1-forms $\{\mathcal{W}_\varepsilon\}$ with the same index is uniformly bounded in L^p_{loc} and that each $\mathcal{W}_\varepsilon$ satisfies the Cartan structural system (2.13) in the sense of distributions. Then, after passing to a subsequence if necessary, $\mathcal{W}_\varepsilon$ converges weakly in L^p_{loc} to a connection 1-form $\mathcal{W}$ that also satisfies system (2.13).*

By "$\{\mathcal{W}_\varepsilon\}$ with the same index" we mean that there are fixed positive integers k and τ such that, for each ε,

$$\mathcal{W}_\varepsilon \in L^p_{\mathrm{loc}}(M; T^*M \otimes \mathfrak{o}(\nu+\tau, (n+k)-(\nu+\tau))).$$

That is, $\{\mathcal{W}_\varepsilon\}$ arises from isometric immersions of M into a fixed semi-Euclidean space $\mathbb{R}^{n+k}_{\nu+\tau}$.

Proof of Theorem 4.1. Our goal is to pass to the limit in the system:

$$\mathrm{d}\mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon \wedge \mathcal{W}_\varepsilon. \tag{4.1}$$

We divide the proof into four steps. Throughout the proof, we write

$$\mathfrak{h} := \mathfrak{o}(\nu+\tau, (n+k)-(\nu+\tau)).$$

1. Take an arbitrary test differential form $\varphi \in C_c^\infty(M\,;\wedge^{n-2}T^*M)$. Then

$$\mathrm{d}\mathcal{W}_\varepsilon \wedge \varphi = \mathcal{W}_\varepsilon \wedge (\mathcal{W}_\varepsilon \wedge \varphi) = \star\langle \star\mathcal{W}_\varepsilon, \mathcal{W}_\varepsilon \wedge \varphi\rangle, \tag{4.2}$$

where $\star : \wedge^j T^*M \to \wedge^{n-j}T^*M$ is the *Hodge star* operator (a vector bundle isomorphism), and φ has no $\mathfrak{h}$-component. In the rest of the proof, we also use $\star$ to denote its natural extension $\star : \wedge^j T^*M\otimes\mathfrak{h} \to \wedge^{n-j}T^*M\otimes\mathfrak{h}$, given by $\star(\omega\otimes A) := \star\omega\otimes A$ for $\omega\in\wedge^jT^*M$ and $A\in\mathfrak{h}$. In other words, we do not distinguish between $\star$ and $\star\otimes\mathrm{id}_{\mathfrak{h}}$.

2. We now determine the differential constraints of Eq. (4.2).

We start from the left-hand side. Notice that $\mathrm{d}\mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon\wedge\mathcal{W}_\varepsilon$ with

$$\mathcal{W}_\varepsilon\wedge\mathcal{W}_\varepsilon \in L^{\frac{p}{2}}_{\mathrm{loc}}(U;\wedge^2\,T^*M\otimes\mathfrak{h}).$$

Recall the following compact Sobolev embedding: If $p<2n$,

$$L^{\frac{p}{2}}_{\mathrm{loc}}(U;\wedge^2\,T^*M\otimes\mathfrak{h}) \hookrightarrow W^{-1,q}_{\mathrm{loc}}(U;\wedge^2\,T^*M\otimes\mathfrak{h}) \qquad \text{for any } q<\frac{pn}{2n-p}.$$

On the other hand, if $p\geq 2n$, we can first embed

$$L^{\frac{p}{2}}_{\mathrm{loc}}(U;\wedge^2\,T^*M\otimes\mathfrak{h}) \to L^{\frac{\hat p}{2}}_{\mathrm{loc}}(U;\wedge^2\,T^*M\otimes\mathfrak{h}) \qquad \text{for } 2<\hat p<2n,$$

and then compactly embed the right-hand side into $W^{-1,q}_{\mathrm{loc}}$. Thus, $\{\mathrm{d}\mathcal{W}_\varepsilon\}$ is pre-compact in $W^{-1,q}_{\mathrm{loc}}(U;\wedge^2T^*M\otimes\mathfrak{h})$ for some $1<q<2$. On the other hand, the Rellich lemma implies that $\{\mathrm{d}\mathcal{W}_\varepsilon\}$ is pre-compact in $W^{-1,p}_{\mathrm{loc}}(U;\wedge^2T^*M\otimes\mathfrak{h})$ for $p>2$. By interpolation, we find that

$$\{\mathrm{d}\mathcal{W}_\varepsilon\} \qquad \text{is pre-compact in } H^{-1}_{\mathrm{loc}}(U;\wedge^2T^*M\otimes\mathfrak{h}).$$

Owing to the super-commutativity of d, we have

$$\mathrm{d}(\mathcal{W}_\varepsilon\wedge\varphi) = \mathrm{d}\mathcal{W}_\varepsilon\wedge\varphi - \mathcal{W}_\varepsilon\wedge\mathrm{d}\varphi.$$

Therefore, we conclude

$$\big\{\mathrm{d}(\mathcal{W}_\varepsilon\wedge\varphi)\big\} \qquad \text{is pre-compact in } H^{-1}_{\mathrm{loc}}(U;\wedge^2\,T^*M\otimes\mathfrak{h}). \tag{4.3}$$

Next, consider the rightmost side of Eq. (4.2). Recall that the L^2-adjoint of d (the co-differential), denoted by $\delta : \wedge^j T^*M \to \wedge^{j-1}T^*M$ for $1 \le j \le n$, is related to d by

$$\delta = (-1)^{j(n-j)+1} \star \mathrm{d} \star .$$

The Hodge star extends to an isometric isomorphism

$$\star \,:\, L^q(U;\wedge^j T^*M) \to L^q(U;\wedge^{n-j}T^*M) \qquad \text{for each } 0 \le j \le n.$$

For M with signature ν,

$$\star\star = (-1)^{j(n-j)+\nu}\, \mathrm{id}_{\wedge^j T^*M},$$

where id denotes the identity map. Then we have obtained another differential constraint:

$$\{\delta \star \mathcal{W}_\varepsilon\} \qquad \text{is pre-compact in } H^{-1}_{\mathrm{loc}}(U;\mathfrak{h}). \tag{4.4}$$

3. In view of the arguments in Step 2 above, especially Eqs. (4.3)–(4.4), it suffices to establish the following *claim*, which is of generality:

Claim: *Let $\{V_\varepsilon\}$ be a family of $(n-1)$-forms so that $\{\mathrm{d}V_\varepsilon\}$ is pre-compact in H^{-1}_{loc}, and let $\{Z_\varepsilon\}$ be a family of $(n-1)$-forms so that $\{\delta Z_\varepsilon\}$ is pre-compact in H^{-1}_{loc}. Assume that $V_\varepsilon \rightharpoonup V$ and $Z_\varepsilon \rightharpoonup Z$ weakly in L^p_{loc}. Then $\{\langle V_\varepsilon, Z_\varepsilon\rangle\}$ converges to $\langle V, Z\rangle$ in the sense of distributions.*

Indeed, if the claim is true, we define

$$\begin{cases} V_\varepsilon := \mathcal{W}_\varepsilon \wedge \varphi \in L^p_{\mathrm{loc}}(U;\wedge^{n-1}T^*M\otimes\mathfrak{h}),\\ Z_\varepsilon := \star\mathcal{W}_\varepsilon \in L^p_{\mathrm{loc}}(U;\wedge^{n-1}T^*M\otimes\mathfrak{h}).\end{cases}$$

The above *claim* implies that $\langle \mathcal{W}_\varepsilon\wedge\varphi, \star\mathcal{W}_\varepsilon\rangle \to \langle \mathcal{W}\wedge\varphi, \star\mathcal{W}\rangle$ in the sense of distributions. Using the identities of the Hodge star and the super-commutativity of the wedge product, we deduce

$$\begin{aligned}\langle \mathcal{W}_\varepsilon \wedge \varphi, \star\mathcal{W}_\varepsilon\rangle &= (-1)^{\nu}\mathcal{W}_\varepsilon \wedge \varphi \wedge \star\star\,\mathcal{W}_\varepsilon\\ &= (-1)^{2\nu+n-1}\mathcal{W}_\varepsilon \wedge \varphi \wedge \mathcal{W}_\varepsilon\\ &= (-1)^{2\nu+2(n-1)}\mathcal{W}_\varepsilon \wedge \mathcal{W}_\varepsilon \wedge \varphi\\ &= \mathcal{W}_\varepsilon \wedge \mathcal{W}_\varepsilon \wedge \varphi.\end{aligned}$$

Therefore, the previous convergence result is equivalent to the following:

$$\mathcal{W}_\varepsilon \wedge \mathcal{W}_\varepsilon \wedge \varphi \longrightarrow \mathcal{W} \wedge \mathcal{W} \wedge \varphi \qquad \text{in the sense of distributions.}$$

Since the test form φ is arbitrary, the proof is now complete.

4. We now prove the *claim* in Step 3 by making crucial use of Theorem 3.2. The key is to specify operator $\mathcal{T}$ and the vector bundles E and F therein.

Indeed, we define

$$\begin{aligned}
E &:= \big(\wedge^{n-1}T^*M \otimes \mathfrak{h}\big) \oplus \big(\wedge^{n-1}T^*M \otimes \mathfrak{h}\big),\\
F &:= \big(\wedge^{n}T^*M \otimes \mathfrak{h}\big) \oplus \big(\wedge^{n-2}T^*M \otimes \mathfrak{h}\big),\\
\mathcal{T} &:= \mathrm{d} \oplus \delta,
\end{aligned}$$

where $\mathcal{T}$ is a bundle operator $\mathcal{T} : E \to F$. In this setting, the operator cone is given by

$$\Lambda_{\mathcal{T}} = \left\{ (\mu,\lambda)^\top \in \Gamma(E) : \begin{array}{c} [\sigma_1(\mathrm{d})(\xi)](\mu) = 0 \text{ and } [\sigma_1(\delta)(\xi)](\lambda) = 0 \\ \text{simultaneously for some } \xi \in T^*M \setminus \{0\} \end{array} \right\},$$

where we have utilized

$$\sigma_1(\mathrm{d}\oplus\delta) = \sigma_1(\mathrm{d}) \oplus \sigma_1(\delta).$$

It is an identity on $\mathcal{P}_1(T^*M; \mathrm{Hom}(E; F^{\mathbb{C}}))$, *i.e.*, the space of first-order homogeneous polynomials that map the cotangent bundle to the homomorphism bundle from E to $F^{\mathbb{C}}$.

We can further specify $\Lambda_{\mathcal{T}}$. Indeed, recall that the principal symbols of d and δ have global intrinsic representations (*cf.* §3.1, [1]):

$$\sigma_1(\mathrm{d})(\xi) = -(2\pi i)\xi\wedge, \qquad \sigma_1(\delta)(\xi) = (2\pi i)\iota_{\xi^\sharp},$$

where $\xi^\sharp$ is the element of the tangent bundle TM canonically isomorphic to ξ (which can be obtained by raising the indices in the local coordinates), and ι_X is the *interior multiplication* of a differential form by the vector field $X \in \Gamma(TM)$. Then

$$\Lambda_{\mathcal{T}} = \left\{ (\mu,\lambda)^\top \in \Gamma(E) : \begin{array}{l} \xi\wedge\mu = 0 \text{ and } \iota_{\xi^\sharp}(\lambda) = 0 \text{ simultaneously} \\ \text{for some } \xi \in T^*M \setminus \{0\} \end{array} \right\}. \tag{4.5}$$

Notice that $\xi \wedge \mu = 0$ if and only if $\mu = (\xi \wedge \tilde{\mu}) \otimes A$ for some $A \in \mathfrak{h}$ and $\tilde{\mu} \in \wedge^{n-2}T^*M$. Also, $\iota_{\xi^\sharp}(\lambda) = 0$ if and only if $\{\tilde{\lambda}, \xi\}$ span an orthogonal subspace in T^*M so that $\lambda = \tilde{\lambda} \otimes B$ for $B \in \mathfrak{h}$.

Now, define the quadratic polynomial $Q : \Gamma(E) \to \mathbb{R}$ by

$$Q((\mu,\lambda)^\top) := \langle \mu, \lambda \rangle.$$

The bracket, $\langle\cdot,\cdot\rangle$, on the right-hand side is the combination of the inner product on $\wedge^{n-1}T^*M$ and the matrix product on $\mathfrak{h}$. Thus, for $(\mu,\lambda)^\top \in \Lambda_{\mathcal{T}}$, we have

$$Q((\mu,\lambda)^\top) = \langle (\xi\wedge\tilde{\mu})\otimes A, \tilde{\lambda}\otimes B\rangle = \langle \xi\wedge\tilde{\mu}, \tilde{\lambda}\rangle \otimes (A\cdot B),$$

where $\cdot$ denotes the matrix multiplication.

Then $\langle \xi\wedge\tilde{\mu}, \tilde{\lambda}\rangle = 0$. Indeed, recall that the dot product $\langle\cdot,\cdot\rangle$ on $\wedge^{n-1}T^*M$ is induced from the inner product on T^*M by the following rule: For two

$(n-1)$-tuples of basic elements in the cotangent bundle T^*M: $\{\theta^{i_1}, \ldots, \theta^{i_{n-1}}\}$ and $\{\theta^{j_1}, \ldots, \theta^{j_{n-1}}\}$, define

$$\left\langle \theta^{i_1} \wedge \ldots \wedge \theta^{i_{n-1}}, \theta^{j_1} \wedge \ldots \wedge \theta^{j_{n-1}} \right\rangle := \det\left(\langle \theta^{i_k}, \theta^{j_l} \rangle_{1 \leq k,l \leq n-1} \right). \tag{4.6}$$

In particular, if some θ^{i_k} is orthogonal to θ^{j_l} in T^*M, then the right-hand side of Eq. (4.6) vanishes. By Eq. (4.5) and the ensuing remark, ξ and $\tilde{\lambda}$ are orthogonal, so that $\langle \xi \wedge \tilde{\mu}, \tilde{\lambda} \rangle = 0$. In effect, we have checked the hypotheses on the operator cone in Theorem 3.2; that is, the quadratic polynomial Q vanishes on cone $\Lambda_{\mathcal{T}}$.

In view of the above arguments, conditions (C-1)–(C-3) in Theorem 3.2 are verified. Applying this theorem, we obtain

$$Q((V_\varepsilon, Z_\varepsilon)^\top) := \langle \mathcal{W}_\varepsilon \wedge \varphi, \star\mathcal{W}_\varepsilon \rangle \longrightarrow \langle \mathcal{W} \wedge \varphi, \star\mathcal{W} \rangle =: Q((V, Z)^\top)$$

in the sense of distributions. Then the *claim* follows, so that the theorem is proved.

The equivalence between the Cartan structural system and the GCR system (Proposition 2.1) implies the weak continuity of the GCR system:

Theorem 4.2 *Let (M, g) be a semi-Riemannian manifold of dimension n with $\mathrm{Ind}(M) = \nu$, $g \in L^\infty_{\mathrm{loc}}(M, O(\nu, n-\nu))$, and the Levi–Civita connection ∇ of g in L^p_{loc} for $p > 2$. Assume that a family of second fundamental forms and normal affine connections $\{(\mathrm{II}_\varepsilon, \nabla^\perp_\varepsilon)\}$ is uniformly bounded in L^p_{loc}, and each $(\mathrm{II}_\varepsilon, \nabla^\perp_\varepsilon)$ satisfies the GCR system (2.8)–(2.10) in the sense of distributions. Then, after passing to a subsequence if necessary, $\{(\mathrm{II}_\varepsilon, \nabla^\perp_\varepsilon)\}$ converges weakly in L^p_{loc} to $(\mathrm{II}, \nabla^\perp)$ that also satisfies Eqs. (2.8)–(2.10).*

As remarked in the introduction, §1, the weak continuity of the Cartan structural and GCR systems (Theorems 4.1–4.2) may alternatively be proved by using the compensated compactness theorems in the Euclidean spaces. For example, the following "generalized div-curl lemma" for wedge products was established as Theorem 1.1 in Robbin–Rogers–Temple [52]:

> *Let $\alpha_\varepsilon \rightharpoonup \alpha$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and let $\beta_\varepsilon \rightharpoonup \beta$ in $L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$, where $\{\alpha_\varepsilon\}, \{\beta_\varepsilon\}, \alpha$, and β are differential forms over $\mathbb{R}^n$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Assume that $\{\mathrm{d}\alpha_\varepsilon\} \subset W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^n; T^*\mathbb{R}^n)$ and $\{\mathrm{d}\beta_\varepsilon\} \subset W^{-1,p'}_{\mathrm{loc}}(\mathbb{R}^n; T^*\mathbb{R}^n)$ are pre-compact. Then $\alpha_\varepsilon \wedge \beta_\varepsilon \to \alpha \wedge \beta$ in the sense of distributions.*

One may apply the above result to deduce Theorem 4.1 by computing in local coordinates and adapting the arguments in Chen–Slemrod–Wang [14]. On the other hand, independent of the goal of proving the $W^{2,p}$ continuity of the GCR and Cartan structural systems, we comment that an extension for the above theorem in $\mathbb{R}^n$ to semi-Euclidean spaces (or more generally, to semi-Riemannian manifolds) appears elusive. It does *not* follow from direct adaptations of the arguments in [52]. Indeed, the proof of [52, Theorem 1.1] relies crucially on the ellipticity of the Laplace–Beltrami operator, for which the following arguments beneath [52, Eq. (4.26), page 616] are central:

From the continuity of Δ^{-1} from $W^{-1,p}(\Omega)$ to $W^{1,p}(\Omega)$[1]*, we conclude*

$$\Delta^{-1}\mathbf{d}\alpha_\varepsilon \in \text{ a compact set in } W^{1,p}(\Omega).$$

However, the Laplace–Beltrami operator Δ on a semi-Riemannian manifold is never elliptic, unless the manifold is Riemannian, so that the arguments in [52] cannot pass through in the semi-Riemannian setting.

To conclude this section, we note that the weak continuity of the GCR and Cartan structural systems (Theorems 4.1–4.2) does not require any assumption on the topology of (M, g).

5 Realization Theorem: From the Cartan Structural Systems to Isometric Immersions of Semi-Riemannian Manifolds

In this section, we address the following problem:

> *Given an n-dimensional semi-Riemannian manifold (M, g) of lower regularity satisfying the GCR system* (*cf. Theorem* 2.1) *in the sense of distributions, seek an isometric immersion $f : (M, g) \hookrightarrow (\mathbb{R}^{n+k}, g_0)$ with the semi-Euclidean metric g_0.*

We refer to it as the *realization problem* — Given a weak solution $(\mathrm{II}, \nabla^\perp)$ to the compatibility equations, we would like to realize it as the geometric data of an isometric immersion.

For a Riemannian manifold M, the realization problem is settled in the affirmative if M is simply-connected. The C^∞ case was proved by Tenenblat [61], and the $W^{2,p}_{\mathrm{loc}}$ case for $p > \dim(M)$ by Mardare [43,44] and Szopos [58]. In [12], we also provided a geometric and intrinsic proof. Although the realization problem for semi-Riemannian manifolds is viewed as a "folklore theorem" (*cf.* Chen [10]), we still find it necessary and non-trivial to give a detailed proof. Indeed, new ideas are required in the following two main points:

(i) the interplay of Cartan's formalism and semi-Riemannian geometry,
(ii) the treatment of manifolds of lower regularity.

5.1 Statement of the Realization Theorem

First of all, we note that the following two conditions are necessary for the realization problem:

(R1) The resulting map f must be an immersion of M as a semi-Riemannian submanifold;

(R2) The indices of manifold M and its normal bundle $TM^\perp = f^*T\mathbb{R}^{n+k}/TM$ (see Convention 2.2) add up to the index of the target space:

$$\mathrm{Ind}(M) + \mathrm{Ind}(T_xM^\perp) = \mathrm{Ind}(\mathbb{R}^{n+k}) \qquad \text{for each } x \in M.$$

[1] There is a typo in [52]: the second $W^{-1,p}(\Omega)$ therein should be $W^{1,p}(\Omega)$.

Indeed, condition (R1) holds since f is an isometry ($f^*g_0 = g$), and a semi-Riemannian metric is non-degenerate by definition. For example, it rules out the possibility that a semi-Riemannian manifold is isometrically embedded into the lightcone of the Minkowski spaces. Condition (R2) is a consequence of (R1) together with the direct sum decomposition in Eq. (2.2).

From now on, we fix the target semi-Euclidean metric to be $\tilde{g}_0$ (defined as in §2.1):

$$\tilde{g}_0 = \mathrm{diag}(\underbrace{-1,\cdots,-1}_{\nu \text{ times}},\ \underbrace{1,\cdots,1}_{n-\nu \text{ times}};\underbrace{-1,\cdots,-1}_{\tau \text{ times}},\ \underbrace{1,\cdots,1}_{k-\tau \text{ times}}), \tag{5.1}$$

and fix $\mathrm{Ind}(M) = \nu$. As before, we write the corresponding semi-Euclidean space as $\mathbb{R}^{n+k}_{\nu+\tau}$.

The main result of this section is Theorem 5.1 below. It gives an affirmative answer to the realization problem of semi-Riemannian manifolds with lower regularity, provided that conditions (R1)–(R2) are satisfied and that the manifold is simply-connected.

Theorem 5.1 *Consider an n-dimensional simply-connected semi-Riemannian manifold (M, g) with metric $g \in W^{1,p}_{\mathrm{loc}}(M; O(\nu, n-\nu))$ for $p > n$ and $\nu = \mathrm{Ind}(M) \in \{0, 1, \cdots, n\}$. Suppose that E is a bundle over M with fiber $F = \mathbb{R}^k_\tau$, bundle metric $g^E \in W^{1,p}_{\mathrm{loc}}(M; O(\tau, k-\tau))$, and bundle connection $\nabla^E \in L^p_{loc}(M; T^*M \otimes \mathrm{End}E)$ compatible with g^E. Let $\mathrm{II} \in L^p_{\mathrm{loc}}(M; \mathrm{Sym}^2 T^*M \otimes E)$ be a symmetric two-tensor, and let S be defined by $g(S_\alpha X, Y) = g^E(\mathrm{II}(X, Y), \alpha)$ for any $X, Y \in \Gamma(TM)$ and $\alpha \in \Gamma(E)$. Moreover, assume that the GCR system on E holds in the sense of distributions. Then there exists a $W^{2,p}_{\mathrm{loc}}$ isometric immersion $f : (M, g) \hookrightarrow (\widetilde{M} = \mathbb{R}^{n+k}_{\nu+\tau}, \tilde{g}_0)$ so that the normal bundle $TM^\perp := f^*T\widetilde{M}/TM$, the second fundamental form, and the shape operator induced by f are identified with E, II, and S, respectively, and f is unique modulo the rigid motions in $(\widetilde{M}, \tilde{g}_0)$.*

In addition, if g, ∇^E, g^E, $\mathrm{II} \in C^\infty$, then there exists a smooth isometric immersion $f \in C^\infty(M; \widetilde{M})$.

Remark 5.1 Concerning the statement of Theorem 5.1, we have

(i) ∇^E is said to be compatible with g^E if, for any $X \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(E)$,
$$Xg^E(\alpha, \beta) = g^E(\nabla^E_X \alpha, \beta) + g^E(\alpha, \nabla^E_X \beta). \tag{5.2}$$
For example, the Levi–Civita connection on M is compatible with g. As in Convention 2.6, we may express Eq. (5.2) as
$$X\langle\alpha, \beta\rangle = \langle\nabla^E_X \alpha, \beta\rangle + \langle\alpha, \nabla^E_X \beta\rangle.$$

(ii) For a bundle E over M, $\mathrm{Sym}^2 E^*$ denotes the space of symmetric 2-tensors defined on E, *i.e.*, each $M \in \Gamma(\mathrm{Sym}^2 E^*)$ satisfies $M(\alpha, \beta) = M(\beta, \alpha)$ for any $\alpha, \beta \in \Gamma(E)$. Note that, in general, a semi-Riemannian metric g on M does not lie in $\Gamma(\mathrm{Sym}^2 T^*M)$. Instead, $g \in \Gamma(O(\nu, n-\nu))$ as $g_{ij}\epsilon^j = g_{ji}\epsilon^i$ (*cf.* §2.1 for the notations).

Remark 5.2 Theorem 5.1 has a global topological consequence as follows: If the GCR equations on the abstract vector bundle E are satisfied under the indicated regularity assumptions, then the trivial rank-$(n+k)$ bundle $T\mathbb{R}^{n+k}_{\nu+\tau}$ has the following Whitney sum decomposition:

$$T\mathbb{R}^{n+k}_{\nu+\tau} = TM \oplus E.$$

Remark 5.3 Theorem 5.1, together with Proposition 2.1, yields the *equivalence* of the following statements, provided that (M, g) is simply-connected and $p > \dim M$:

(i) The existence of isometric immersions of semi-Riemannian manifolds;

(ii) The solvability of the GCR system in the sense of distributions;

(iii) The solvability of the Cartan structural system in the sense of distributions.

5.2 Proof of the Realization Theorem, Theorem 5.1

If everything is C^∞, then the Frobenius theorem on the equivalence of involutive and completely integrable distributions can be directly applied, and hence we may adapt the proof by Tenenblat [61] for the smooth Riemannian case. In the case of lower regularity, we only need to replace the Frobenius theorem with an analogous existence and regularity theorem for certain first-order PDE systems with Sobolev coefficients.

Proof of Theorem 5.1. Without loss of generality, we can first assume the result holds for the C^∞ case. As remarked above, to this end, we can adapt Tenenblat's arguments in [61], taking into account various modifications required by non-trivial signatures in the semi-Riemannian setting. See Appendix A.5 for the details of the proof.

Now we show for the lower regularity case: $g \in W^{1,p}_{\mathrm{loc}}(M, O(\nu, n-\nu))$. As in Appendix A.5, assume that the *Pfaff* and *Poincaré* systems with

$$g \in W^{1,p}_{\mathrm{loc}}(M, O(\nu, n-\nu)), \quad \mathcal{W} \in L^p_{\mathrm{loc}}(U \subset M; \mathfrak{o}(\nu+\tau, (n+k)-(\nu+\tau)))$$

are solved; that is, there exist a bundle connection A and an immersion f in the following spaces:

$$\begin{cases} A \in W^{1,p}_{\mathrm{loc}}(U \subset M; O(\nu+\tau, (n+k)-(\nu+\tau))), \\ f \in W^{2,p}_{\mathrm{loc}}(M; \widetilde{M}), \end{cases}$$

such that $\operatorname{rank}(\mathrm{d}f) = n$. Then f is indeed an $W^{2,p}_{\mathrm{loc}}$ isometric immersion by construction. The Pfaff and Poincaré systems are, respectively, as follows:

$$\mathcal{W} = \mathrm{d}A \cdot A^{-1}, \qquad A(0) = A(x_0), \tag{5.3}$$

and

$$\mathrm{d}f = \underset{\sim}{\Theta} \cdot A, \qquad f(0) = f(x_0), \tag{5.4}$$

where x_0 is a given point in a local chart $U \subset M$.

The solvability of the Poincaré system (5.4) with Sobolev coefficients is easy to be established. For any given

$$A \in W^{1,p}_{\mathrm{loc}}(U \subset M; O(\nu+\tau, (n+k)-(\nu+\tau))),$$

we want to solve for f in $W^{2,p}_{\mathrm{loc}}(M; \widetilde{M})$. Since all the results are stated in local Sobolev spaces, it suffices to assume that U is a smooth bounded open subset of $\mathbb{R}^n$. In this setting, choose $J_\varepsilon \in C^\infty(\mathbb{R}^n)$ to be the standard mollifier and set $\Theta_\varepsilon := J_\varepsilon * (\underset{\sim}{\Theta} \cdot A)$. It follows that

$$\Theta_\varepsilon \longrightarrow \underset{\sim}{\Theta} \cdot A \qquad \text{in } W^{1,p}(U; \widetilde{M}) \quad \text{as } \varepsilon \to 0^+.$$

In particular, $\{\Theta_\varepsilon\}$ is uniformly bounded in $W^{1,p}$.

Now, Θ_ε is a smooth closed 1-form (*cf.* Appendix A.5) for each $\varepsilon > 0$, so we can invoke the solvability of the Poincaré system in the C^∞ case to find some $f_\varepsilon \in C^\infty(U; \widetilde{M})$ with $\mathrm{d}f_\varepsilon = \Theta_\varepsilon$. By adding a constant, we may assume that $\int_U f_\varepsilon \, \mathrm{d}x = 0$. Then the Poincaré inequality gives us

$$\|f_\varepsilon\|_{W^{2,p}(U;\widetilde{M})} \leq C\big(\|f_\varepsilon\|_{W^{1,p}(U;\widetilde{M})} + \|\Theta_\varepsilon\|_{W^{1,p}(U;\widetilde{M})}\big).$$

Hence, thanks to the Rellich lemma and the uniform boundedness of $\{\Theta_\varepsilon\} \subset W^{1,p}(U; \widetilde{M})$, we obtain that $\|f_\varepsilon\|_{W^{2,p}(U;\widetilde{M})} \leq C_0 < \infty$. Therefore, there exists a limiting function $\tilde{f}$ so that $f_\varepsilon \to \tilde{f}$ in $W^{2,p}(U; \widetilde{M})$ (modulo subsequences) with $\mathrm{d}\tilde{f} = \underset{\sim}{\Theta} \cdot A$.

The Pfaff system (5.3) with Sobolev coefficients is more difficult to tackle: The Frobenius theorem cannot be directly applied, since we need at least C^1–regularity; in addition, we cannot apply a simple mollification argument, since the compatibility condition (*i.e.*, the second structural system $\mathrm{d}\mathcal{W} = \mathcal{W} \wedge \mathcal{W}$) contains quadratic nonlinear terms.

However, the following result serves for our purpose:

Lemma 5.1 (Mardare [44]) *Let $\Omega \subset \mathbb{R}^n$ be a simply-connected open set, $x_0 \in \Omega$, and $M_0 \in \mathfrak{gl}(l; \mathbb{R})$. Then the following system*:

$$\frac{\partial M}{\partial x^i} = \mathfrak{S}_i \cdot M, \quad i = 1, 2, \ldots, n, \qquad M(x_0) = M_0,$$

with the matrix fields $\mathfrak{S}_i \in L^p_{\mathrm{loc}}(\Omega; \mathfrak{gl}(l; \mathbb{R}))$ for $i = 1, 2, \ldots, n$, and $p > n$, has a unique solution $M \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathfrak{gl}(l; \mathbb{R}))$ if and only if the following compatibility condition holds:

$$\frac{\partial \mathfrak{S}_i}{\partial x^j} - \frac{\partial \mathfrak{S}_j}{\partial x^i} = [\mathfrak{S}_i, \mathfrak{S}_j] \qquad \textit{for each } i, j = 1, 2, \ldots, n, \tag{5.5}$$

in the sense of distributions.

As Lemma 5.1 is formulated for $\Omega \subset \mathbb{R}^n$, we correspondingly take $U \subset M$ as a trivialized local chart so that bundle E can be regarded as $U \times \mathbb{R}^k$ over U. Hence, on U, without loss of generality, we may assume that $[\partial_i, \partial_j] = 0$. We take

$$\mathfrak{S} = \mathcal{W} \in L^p_{\rm loc}(U; T^*M \otimes \mathfrak{o}(\nu+\tau, (n+k)-(\nu+\tau))), \qquad \mathfrak{S}_i = \mathcal{W}(\partial_i).$$

Then

$$\partial_i \mathfrak{S}_j - \partial_j \mathfrak{S}_i = \partial_i(\mathcal{W}(\partial_j)) - \partial_j(\mathcal{W}(\partial_i)) + \mathcal{W}([\partial_i, \partial_j]) = {\rm d}\mathcal{W}(\partial_i, \partial_j).$$

On the other hand, we have

$$[\mathfrak{S}_i, \mathfrak{S}_j] = \mathcal{W}(\partial_i) \cdot \mathcal{W}(\partial_j) - \mathcal{W}(\partial_j) \cdot \mathcal{W}(\partial_i) = (\mathcal{W} \wedge \mathcal{W})(\partial_i, \partial_j).$$

Thus, the compatibility condition in Lemma 5.1 is verified by the second structural system (2.13). The Pfaff system (5.3) with Sobolev coefficients is hence uniquely solvable on local charts.

Therefore, we now arrive at the existence of a local isometric immersion in the lower regularity case, provided that the second structural system (or equivalently, the GCR system) holds in the sense of distributions.

Finally, we deduce the global existence of an isometric immersion, which follows from a standard monodromy argument. Given any two points $x, y \in M$ with $x \neq y$, we connect them by a continuous curve (again since $W^{1,p}_{\rm loc} \hookrightarrow C^0_{\rm loc}$ for $p > n$), denoted by $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. More precisely, γ is chosen as a continuous representative in the Sobolev space. Let f be the $W^{2,p}_{\rm loc}$ isometric immersion in a neighborhood of x, whose existence is guaranteed by the earlier steps. We cover $\gamma([0,1])$ by finitely many charts $\{V^1, \ldots, V^N\}$. By the uniqueness statement in Lemma 5.1, we can extend the isometric immersion f to $\bigcup_{i=1}^N V^i$, especially including a neighborhood of y.

Thus, it suffices to show that the extension of f is independent of the choice of γ. Indeed, if $\eta : [0,1] \to M$ is another continuous curve connecting x and y, by concatenating γ with η, we form a loop $L \subset M$. As M is simply-connected, the restriction $f|_L$ is homotopic to a constant map so that $(f \circ \gamma)(1) = (f \circ \eta)(1)$. In this way, we have verified that f can be extended to a global isometric immersion of M into $\widetilde{M}$, provided that M is simply-connected. This completes the proof.

As a remark, in the realization theorem, Theorem 5.1, it requires that $g \in W^{1,p}_{\rm loc}$ with $p > n = \dim M$. This is because of both the regularity assumptions in Lemma 5.1 and the continuity requirements for the topological arguments. All the other results in this paper hold for $p > 2$, regardless of the dimension of M. Also note that (M, g) is assumed to be simply-connected in Theorem 5.1, which prevents the occurrence of branched immersions.

5.3 Weak Rigidity of Isometric Immersions of Semi-Riemannian Manifolds

Recall that, in Theorem 4.1, we have established the weak continuity of the Cartan structural system on a semi-Riemannian (M, g) and, in Proposition 2.1, we have shown the equivalence of the structural system with the GCR system, both for $p > 2$ regardless of $\dim M$. If we translate this PDE-theoretic weak continuity theorem into geometric settings, then it is unsurprising that the $W^{2,p}_{\mathrm{loc}}$ isometric immersions of M are weakly rigid. More precisely, we have

Theorem 5.2 *Let (M, g) be a semi-Riemannian manifold of dimension n with $\mathrm{Ind}(M) = \nu$, $g \in L^\infty_{\mathrm{loc}}(M; O(\nu, n-\nu))$, and the Levi–Civita connection ∇ of g in L^p_{loc} for $p > 2$. Let $\{f_\varepsilon\} \subset W^{2,p}_{\mathrm{loc}}(M; \mathbb{R}^{n+k})$ be a family of isometric immersions of semi-Riemannian submanifolds, with the second fundamental forms $\{\mathrm{II}_\varepsilon\}$ and normal connections $\{\nabla^\perp_\varepsilon\}$ satisfying GCR system* (2.8)–(2.10). *Assume that $\{f_\varepsilon\}$ is uniformly bounded in $W^{2,p}_{\mathrm{loc}}$ and $\mathbb{R}^{n+k}$ is endowed with the semi-Euclidean metric $\tilde{g}_0$ as in Eq.* (5.1). *Then, after passing to a subsequence if necessary, $\{f_\varepsilon\}$ weakly converges in $W^{2,p}_{\mathrm{loc}}$ to an isometric immersion $f \in W^{2,p}_{\mathrm{loc}}(M; \mathbb{R}^{n+k})$; in addition, the second fundamental form and the normal connection of f are the weak L^p_{loc} limits of $\{\mathrm{II}_\varepsilon\}$ and $\{\nabla^\perp_\varepsilon\}$, respectively, and still satisfy the GCR system.*

The same result holds if $\{(\mathrm{II}_\varepsilon, \nabla^\perp_\varepsilon)\}$ are replaced by the connection 1-*forms $\{\mathcal{W}_\varepsilon\}$, and the GCR system is replaced by the Cartan structural system* (2.13).

Proof Let $\{f_\varepsilon\}$ be a bounded family in $W^{2,p}_{\mathrm{loc}}$ where $p > 2$. Then, modulo subsequences, $\{\mathrm{d}f_\varepsilon\}$ is weakly convergent in $W^{1,p}_{\mathrm{loc}}$, hence strongly convergent in L^p_{loc} due to the Rellich lemma. Thus, after passing to a subsequence and thanks to the Hölder inequality, $\tilde{g}_0(\mathrm{d}f_\varepsilon, \mathrm{d}f_\varepsilon)$ converges strongly in $L^{\frac{p}{2}}_{\mathrm{loc}}$ to $\tilde{g}_0(\mathrm{d}\tilde{f}, \mathrm{d}\tilde{f})$, which equals to metric g by assumption, where $\tilde{f}$ is a weak $W^{2,p}_{\mathrm{loc}} \cap W^{1,\infty}_{\mathrm{loc}}$ limit of $\{f_\varepsilon\}$. In addition, by passing to a further subsequence, we may deduce that $\mathrm{d}f_\varepsilon \to \mathrm{d}\tilde{f}$ *a.e.* from the strong L^p_{loc} convergence and that $\tilde{g}_0(\mathrm{d}\tilde{f}, \mathrm{d}\tilde{f}) = g$ *a.e.* from the strong $L^{\frac{p}{2}}_{\mathrm{loc}}$ convergence, by virtue of $p > 2$. This shows that $\tilde{f}$ is an isometric immersion, again in the *a.e.* sense.

On the other hand, by the weak continuity of the GCR system in Theorem 4.1, we find that the second fundamental form and the normal connection of the limiting isometric immersion $\tilde{f}$ — which are weak L^p_{loc} limits of the related quantities for f_ε (possibly modulo a further subsequence) — satisfy the GCR equations in the sense of distributions. This observation together with Proposition 2.1 completes the proof.

In the case that $p > n$, the above result follows directly from the realization theorem (Theorem 5.1), together with Theorem 4.1 and Proposition 2.1. In fact, it can be proved easily for $p > n$ without applying any of the machineries above, but just using the Sobolev-Morrey embedding $W^{2,p}_{\mathrm{loc}} \hookrightarrow W^{1,\infty}_{\mathrm{loc}}$ and the identity $\mathrm{II}_{jk} = \partial_j \partial_k f - \Gamma^i_{jk} \partial_i f$ (see, *e.g.*, Bryant–Griffith–Yang [8, page 959] for the Riemannian case). The main point of our arguments here is to extend to the case $p > 2$, irrespective of dim M.

In particular, we comment that, *under the stronger hypotheses* that both M is simply-connected and $p > n = \dim M$, Theorems 4.1–4.2 can be deduced easily from the realization theorem (Theorem 5.1), in view of Remark 5.3.

Alternative Proof for Theorem 4.1–4.2 **with** $\pi_1(M) = \{0\}$ **and** $p > n$.

Without loss of generality, we may assume that M is compact and that f_ε converges weakly in $W^{2,p}$ to a map $f : M \to \mathbb{R}^{n+k}_{\nu+\tau}$. Since the embedding $W^{1,p} \hookrightarrow C^0$ is now compact for $p > n$, by choosing continuous representatives in suitable Sobolev classes, $g_\varepsilon := f_\varepsilon^* \tilde{g}_0$ converges uniformly to $g := f^* \tilde{g}_0 \in W^{1,p}$.

Note that $f_\varepsilon : (M, g_\varepsilon) \hookrightarrow (\mathbb{R}^{n+k}_{\nu+\tau}, \tilde{g}_0)$ and $f : (M, g) \hookrightarrow (\mathbb{R}^{n+k}_{\nu+\tau}, \tilde{g}_0)$ are isometric immersions by construction. By the realization theorem, Theorem 5.1, the connection 1-forms $\mathcal{W}_\varepsilon$ and $\mathcal{W}$ (corresponding to f_ε and f, respectively) satisfy the Cartan structural systems:

$$\mathrm{d}\mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon \wedge \mathcal{W}_\varepsilon, \qquad \mathrm{d}\mathcal{W} = \mathcal{W} \wedge \mathcal{W}.$$

These two systems are well-defined, with the left-hand sides in $W^{1,p}$ and the right-hand sides in $L^{\frac{p}{2}}$ for $p > n \geq 2$. Also, Definition 2.8 for the connection 1-forms implies that $\mathcal{W}_\varepsilon \rightharpoonup \mathcal{W}$ in L^p. Then Theorem 4.1 follows when $\pi_1(M) = \{0\}$ and $p > n$. We can conclude Corollary 4.2 from Proposition 2.1.

Nonetheless, we emphasize once more that the above short proof is available *only for* $p > n$; the argument does not extend to the less stringent case $p > 2$, even with Theorem 5.2 at hand. This is because the current proof of the realization theorem (Theorem 5.1; *cf.* also Szopos [58]) essentially needs $p > n$, as it is crucial for Lemma 5.1.

6 Further Applications

In this final section, we present some further applications of the results and techniques developed in §2–§5 above.

(i) Using the weak continuity of isometric immersions (Theorem 5.2), we show the weak continuity of the constraint equations in general relativity;
(ii) Directly utilizing the geometric compensated compactness theorem, Theorem 3.2, we establish the weak continuity of quasilinear wave equations satisfying the *null condition* (introduced first by Klainerman [37]; see also [22,38]).
(iii) Employing a generalized version of the GCR system, we prove the weak continuity of general immersed hypersurfaces, *i.e.*, the 1-co-dimensional submanifolds with possibly degenerate induced metrics.

6.1 Weak Rigidity of Einstein's Constraint Equations

Let (V, g) be a Lorentzian manifold of dimension $N+1$. The vacuum Einstein field equation is

$$\mathrm{Ric}_g = 0,$$

that is, the Ricci curvature of g vanishes. This system consists of $\frac{(N+1)(N+2)}{2}$ scalar equations, in which $N+1$ equations are determined by the initial data on some space-like hypersurface via the Gauss–Codazzi equations. These $N+1$ equations are known as Einstein's constraint equations; see Bartnik–Isenberg [4], Choquet–Bruhat [16], Corvino–Schoen [23], and the references cited therein.

In the Minkowski case $(\mathbb{R}^{N+1}, \mathfrak{m})$, we can show the following theorem:

Theorem 6.1 *Let M be a space-like hypersurface of the Minkowski space-time $(\mathbb{R}^{N+1}, \mathfrak{m})$ with a family of immersions $\{f_\varepsilon\}$. Denote by $\gamma_\varepsilon := f_\varepsilon^* \mathfrak{m}$ the pull-back metrics on M. Suppose that, for each fixed $\varepsilon > 0$, (M, γ_ε) satisfies the Einstein constraint equations in the vacuum*:

$$\begin{cases} \mathrm{scal}_\varepsilon + (\mathrm{tr}_{\gamma_\varepsilon} h_\varepsilon)^2 - |h_\varepsilon|^2 = 0, \\ \sum_{j=1}^N \widetilde{\nabla}^j \big((h_\varepsilon)_{ij} - \mathrm{tr}_{\gamma_\varepsilon}(h_\varepsilon)(\gamma_\varepsilon)_{ij}\big) = 0 \qquad \textit{for } i = 1, 2, \ldots, n. \end{cases} \tag{6.1}$$

In the above, $\widetilde{\nabla}$ is the Levi–Civita connection on $(\mathbb{R}^{N+1}, \mathfrak{m})$, $\mathrm{scal}_\varepsilon$ is the scalar curvature of (M, γ_ε), and h_ε is the second fundamental form:

$$\widetilde{\nabla}_X Y = (\nabla_\varepsilon)_X Y + h_\varepsilon(X, Y)\mathbf{n}_\varepsilon \qquad \textit{for all } X, Y \in \Gamma(TM),$$

where ∇_ε is the Levi–Civita connection on (M, γ_ε) and $\mathbf{n}_\varepsilon$ is the time-like unit normal. If $\{f_\varepsilon\}$ is uniformly bounded in $W^{2,p}_{\mathrm{loc}}(M, \mathbb{R}^{N+1})$ for $p > 2$, then it converges weakly in $W^{2,p}_{\mathrm{loc}}$ to an immersion $\tilde{f} : M \to (\mathbb{R}^{N+1}, \mathfrak{m})$ such that $(M, \tilde{f}^* \mathfrak{m})$ satisfies Einstein's constraint equations in the sense of distributions.

Proof By construction, $f_\varepsilon : (M, \gamma_\varepsilon) \to (\mathbb{R}^{N+1}, \mathfrak{m})$ is an isometric immersion for each $\varepsilon > 0$. Then $f_\varepsilon \rightharpoonup \tilde{f}$ in $W^{2,p}_{\mathrm{loc}}$, where $\tilde{f}$ is an isometric immersion whose second fundamental form satisfies the Gauss–Codazzi equations in the sense of distributions, by Theorem 5.2. However, the constraint equations (6.1) are implied by the Gauss–Codazzi equations (see Bartnik–Isenberg [4]). In view of Remark 5.3, the assertion now follows.

6.2 Weak Continuity of Quasilinear Wave Equations

Now we give an application of our quadratic theorem of compensated compactness, *i.e.*, Theorem 3.2, to the weak continuity of a special class of nonlinear wave equations:

$$\Box_{\mathfrak{m}} \phi^I = F^I(\phi, \partial\phi) \qquad \text{for all } I \in \{1, 2, \ldots, N\}. \tag{6.2}$$

This system is posed on $(\mathbb{R}^{3+1}, \mathfrak{m})$, where $\mathfrak{m} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric, and $F = \{F^I\}_{1\leq I\leq N}$ is the source function. We are concerned with $\phi = \{\phi^I\}_{1\leq I\leq N} : \mathbb{R}^{3+1} \to \mathbb{R}^N$. The source function F consists of quadratic terms with respect to $(\phi, \partial\phi)$, where ∂ denotes the total space-time derivative.

A classical result due to Christodoulou [22] and Klainerman [38] is the following: When the smooth initial data is sufficiently small, the Cauchy problem for Eq. (6.2) has a unique solution $\phi \in C_c^\infty([0,\infty)\times\mathbb{R}^3;\mathbb{R}^N)$, provided that F satisfies the *null condition*:

(i) $F^I(0) = 0$ and $\partial F^I(0) = 0$,

(ii) $Q_{F^I}(\partial\phi) = \sum_{J,K=1}^N \sum_{\mu,\nu=0}^3 A^{\mu\nu}_{IJK}(\partial_\mu\phi^J)(\partial_\nu\phi^K)$ for each $I \in \{1,2,\ldots,N\}$ with

$$\sum_{\mu,\nu=0}^3 A^{\mu\nu}_{IJK}\xi_\mu\xi_\nu = 0$$

for any null co-vector $\xi \in T^*\mathbb{R}^{3+1}$ and $I,J,K \in \{1,2,\ldots,N\}$, where Q_{F^I} denotes the quadratic part in $\partial\phi$ in the Taylor expansion of F^I at $(\phi,\partial\phi) = 0$:

$$Q_{F^I}(z) := \sum_{|\alpha|=2} \frac{\partial_\alpha F^I(0)}{\alpha!} z^\alpha \qquad \text{for all } z \in \mathbb{R}^N$$

in the multi-index notations, and $\xi \in T^*\mathbb{R}^{3+1}$ is a null co-vector if and only if $\mathfrak{m}^{\mu\nu}\xi_\mu\xi_\nu = 0$.

For our purpose, we take the following bundle of type–$(1,1)$ tensors:

$$E = T^*\mathbb{R}^{3+1} \otimes T\mathbb{R}^N.$$

Then, for each $I \in \{1,2,\ldots,N\}$, define the bundle operator $\mathcal{T}_I \in \mathrm{Hom}(E;\mathbb{R})$:

$$\mathcal{T}_I\, s := \sum_{J,K=1}^N \sum_{\mu,\nu=0}^3 A^{\mu\nu}_{IJK}(\partial_\nu s^J_\mu)\theta^K, \tag{6.3}$$

where $\{\theta^K\} \subset T^*\mathbb{R}^N$ is the co-vector basis dual to $\{\partial_K\}$. The associated operator cone is

$$\Lambda_{\mathcal{T}_I} := \left\{ \lambda \in T^*\mathbb{R}^{3+1}\otimes T\mathbb{R}^N \; : \; \begin{array}{l} \sum_{\mu,\nu=0}^3 A^{\mu\nu}_{IJK}\lambda^J_\mu s^K_\nu = 0 \text{ for some non-zero} \\ \text{section } s \in \Gamma(T^*\mathbb{R}^{3+1}\oplus T\mathbb{R}^N)\setminus\{0\} \end{array} \right\}. \tag{6.4}$$

The following observation is crucial: For each null co-vector $\xi \in T^*\mathbb{R}^{3+1}$, if it is identified with $\xi\otimes\mathrm{id} \in T^*\mathbb{R}^{3+1}\otimes T\mathbb{R}^N$ (where id is the tautological tensor on $T\mathbb{R}^N$), then it lies in $\Lambda_{\mathcal{T}_I}$. In other words, the *null cone* of the spacetime $(\mathbb{R}^{3+1},\mathfrak{m})$ can be viewed as a subset of the operator cone $\Lambda_{\mathcal{T}_I}$ for every $I \in \{1,2,\ldots,N\}$.

Also, for each $I \in \{1, 2, \ldots, N\}$, consider the quadratic form:

$$Q_{F^I}(s) := \sum_{J,K=1}^{N} \sum_{\mu,\nu=0}^{3} A^{\mu\nu}_{IJK} s^J_\nu s^K_\mu \quad \text{for } s = \{s^J_\mu\}_{1\leq J\leq N, 0\leq\mu\leq 3} \in \Gamma(E). \tag{6.5}$$

It can be defined intrinsically on $\Gamma(E)$. It is easy to check that Q_{F^I} agrees with the quadratic terms in $\partial\phi$ of the source term F^I.

Now, applying Theorem 3.2 to the sequence of sections

$$\{\partial\phi_\varepsilon\} \subset L^2_{\mathrm{loc}}(\mathbb{R}^{3+1}; T^*\mathbb{R}^{3+1} \otimes T\mathbb{R}^N),$$

we obtain the following compensated compactness framework, which enables us to verify the H^1_{loc} weak continuity of Eq. (6.2). Indeed, it requires to pass the limits in the source term $F^I(\phi_\varepsilon, \partial\phi_\varepsilon)$, as the left-hand side of the equation is linear in ϕ_ε.

Proposition 6.1 *Let the source term $F^I(\phi, \partial\phi)$ satisfy the null condition so that*

$$Q_{F^I}(s) = 0 \qquad \text{for any } s \in \Lambda_{\mathcal{T}^I}, \tag{6.6}$$

where the operator cone $\Lambda_{\mathcal{T}^I}$ is defined according to Eqs. (6.4)–(6.5)*. Assume that $\{\phi_\varepsilon\}$ is a family of functions in $H^1_{\mathrm{loc}}(\mathbb{R}^{3+1}, \mathbb{R}^N)$ such that*

(i) $\phi_\varepsilon \rightharpoonup \phi$ *weakly in* H^1_{loc};

(ii) $\big\{\sum_{J=1}^N \sum_{\mu,\nu=0}^3 A^{\mu\nu}_{IJK} \partial_\mu\partial_\nu\phi^J_\varepsilon\big\}$ *is pre-compact in* $H^{-1}_{\mathrm{loc}}(\mathbb{R}^{3+1})$ *for all* $I, K \in \{1, 2, \ldots, N\}$.

Then

$$Q_{F^I}(\partial\phi_\varepsilon) \rightharpoonup Q_{F^I}(\partial\phi) \qquad \text{in the sense of distributions.}$$

As a consequence, if Eq. (6.2) *admits a family of weak solutions $\{\phi_\varepsilon\} \subset H^1_{\mathrm{loc}}(\mathbb{R}^{3+1}, \mathbb{R}^N)$ satisfying* (i)–(ii)*, then the weak limit ϕ in H^1 is also a weak solution of* (6.2)*.*

In particular, a necessary condition for (6.6) above is that $Q_{F^I}(\xi \otimes \mathrm{id}) = 0$ for any null co-vector $\xi \in T^*\mathbb{R}^{3+1}$.

The above proposition shows that the quasilinear wave equation with null condition in $3 + 1$ dimensions is weakly continuous in H^1_{loc}. However, it is well-known (*cf.* Rodnianski [53]) that the Einstein equations fail to satisfy the null conditions, even in the vacuum or scalar field cases. It would be interesting to analyze further the weak continuity of the Einstein equations and other physical/geometric PDEs.

6.3 Weak Rigidity of General Immersed Hypersurfaces

We now discuss the weak rigidity of immersed hypersurfaces that are *not* semi-Riemannian submanifolds of the ambient spaces. It is remarked in §5.1 (*cf.* Condition (R1)) that, if metric $\tilde{g}_0$ is degenerate on a hypersurface Σ, then Σ cannot be obtained via an isometric immersion of any semi-Riemannian manifold. Nevertheless, such degenerate scenarios occur naturally in physics.

One primary example is the *lightcone*:

$$\Lambda = \{(t, x_1, x_2, x_3) \in \mathbb{R}^4 \,:\, t^2 = x_1^2 + x_2^2 + x_3^2\}$$

of the Minkowski space-time $(\mathbb{R}^{3+1}, \mathfrak{m})$ with $\mathfrak{m} = \mathrm{diag}(-1,1,1,1)$. Although, for any $x, v \in \Lambda$, $g_x(v, w) \neq 0$ for all time-like vectors w in the tangent space at x, we see that $g_x(v, \cdot) \equiv 0$ on $T_x\Lambda$, where Λ is known as a *null hypersurface*. In addition, the stationary limit surface of Kerr's vacuum solution is everywhere time-like, except at the points on the axis where it is null and tangent to the horizon (*cf.* [45]). A more recent example in [46] is the gluing of two Anti-de-Sitter (AdS) 5-dimensional space-times with different cosmological constants along a general hypersurface $\Sigma = \Sigma^E \sqcup \Sigma^{\rm null} \sqcup \Sigma^L$, where $\Sigma^{\rm null}$ is 3-dimensional, such that the restriction of the metric is time-like on Σ^L, space-like on Σ^E, and null on S. If the coordinate system is suitably chosen, $\Sigma^{\rm null}$ may lie in the hypersurface of form $\{t = t_0\}$. This example gives a possible model for the transition between two distinct AdS universes across *brane* Σ, whence $\Sigma^{\rm null}$ models the big-bang singularity.

Motivated by the physical applications above, a treatment for the realization problem and the weak rigidity of general hypersurfaces is desired. However, the constructions in §2.2, especially the derivation of the GCR system or the Cartan structural system, fail in this case — the orthogonal decomposition of tangent spaces as in Eq. (2.1) is no longer valid.

To overcome this difficulty, we employ the construction of *rigging vector fields*; *cf.* [40,41,45,54]. The idea is as follows: Consider the hypersurface via the local embedding $\iota : \Sigma \hookrightarrow (\widetilde{M}, \tilde{g})$. If $\iota^*\tilde{g}$ is null, we find a non-vanishing vector field $\ell \in \Gamma(\iota^*T\widetilde{M})$ along Σ so that

$$T_{\iota(x)}\widetilde{M} \cong T_x\Sigma \oplus \mathrm{span}\{\ell_x\}. \tag{6.7}$$

Thus, we can derive the Gauss–Codazzi equations (for hypersurfaces, the Ricci equation is always trivial) from the orthogonal decomposition (6.7). However, technicalities are unavoidable because the rigging field ℓ *never* coincides with the normal vector field, whenever Σ is null — this leads to three Codazzi equations instead of one.

From now on, α always denotes a co-vector field, *i.e.*, an element of $\Gamma(T^*\Sigma)$. This is in agreement with [45,54]. The first main result in this subsection is

Theorem 6.2 *Let $\iota : \Sigma \hookrightarrow (\mathbb{R}^{n+1}, \tilde{g}_0)$ be a $W^{2,p}_{\rm loc}$ immersion of a simply-connected general hypersurface for $p > n$, for which the pullback tensor $\iota^*\tilde{g}_0$ is* *allowed to degenerate on Σ. Let the normal* 1*-form of Σ to be $\mathfrak{n} \in \Gamma(\iota^* T^*\mathbb{R}^{n+1})$. Moreover, assume that $\ell \in \Gamma(T\mathbb{R}^{n+1})$ is a rigging vector field, i.e., $\mathfrak{n}(\ell) = 1$ everywhere on Σ. Take $\{e^i\}_{i=1}^n \subset \Gamma(T\Sigma)$ as an orthonormal frame on Σ, and $\{\theta^i\}_{i=1}^n \subset \Gamma(T^*\Sigma)$ as its co-frame. Furthermore, define the tensor fields $K \in W^{1,p}_{\mathrm{loc}}(\Sigma; \wedge^2 T^*\Sigma)$ and $\Psi \in W^{1,p}_{\mathrm{loc}}(\Sigma; T\Sigma \otimes T^*\Sigma) = W^{1,p}_{\mathrm{loc}}(\Sigma; \mathrm{End}\, T\Sigma)$ by*

$$K := \widetilde{\nabla}\mathfrak{n}, \qquad \Psi := \widetilde{\nabla}\ell,$$

that is,

$$K(X,Y) = \widetilde{\nabla}\mathfrak{n}(X,Y), \qquad \Psi(\alpha, X) := \alpha(\widetilde{\nabla}_X \ell)$$

for each $X, Y \in \Gamma(T\Sigma)$ and $\alpha \in \Gamma(T^*\Sigma)$. Define also $\psi \in W^{1,p}_{\mathrm{loc}}(\Sigma; T^*\Sigma)$ by

$$\psi(X) := \mathfrak{n}(\widetilde{\nabla}_X \ell).$$

Then the following equations hold in the sense of distributions:

$$\alpha(R(X,Y)Z) - K(Y,Z)\Psi(\alpha,X) + K(X,Z)\Psi(\alpha,Y) = 0, \tag{6.8}$$

$$\begin{aligned} &K(X, \nabla_Y Z) - K(Y, \nabla_X Z) + K([X,Y], Z) + XK(Y,Z) - YK(X,Z) \\ &\quad - K(Y,Z)\psi(X) + K(X,Z)\psi(Y) = 0, \end{aligned} \tag{6.9}$$

$$\begin{aligned} &X\Psi(\alpha,Y) - Y\Psi(\alpha,X) + \Psi(\alpha,[X,Y]) + \psi(Y)\Psi(\alpha,X) - \psi(X)\Psi(\alpha,Y) \\ &\quad + \sum_{i=1}^n \big\{\Psi(\theta^i, Y)\alpha(\nabla_X e^i) - \Psi(\theta^i, X)\alpha(\nabla_Y e^i)\big\} = 0, \end{aligned} \tag{6.10}$$

$$\begin{aligned} &X\psi(Y) - Y\psi(X) + \psi([X,Y]) \\ &\quad + \sum_{i=1}^n \big\{K(e^i, Y)\Psi(\theta^i, X) - K(e^i, X)\Psi(\theta^i, Y)\big\} = 0 \end{aligned} \tag{6.11}$$

for $X, Y, Z \in \Gamma(T\Sigma)$ and $\alpha \in \Gamma(T^*\Sigma)$ such that $\alpha(l) = 0$, and R is the Riemann curvature of Σ.

Conversely, if Eqs. (6.8)–(6.11) *hold in the sense of distributions for $K \in W^{1,p}_{\mathrm{loc}}(\Sigma; \wedge^2 T^*\Sigma)$, $\Psi \in W^{1,p}_{\mathrm{loc}}(\Sigma; \mathrm{End}\, T\Sigma)$, and $\psi \in W^{1,p}_{loc}(\Sigma; T^*\Sigma)$, then there exist an immersion $\iota \in W^{2,p}_{\mathrm{loc}}(\Sigma; \mathbb{R}^{n+1})$ and a rigging vector field $\ell \in \Gamma(T\mathbb{R}^{n+1})$ such that $K = \widetilde{\nabla}\mathfrak{n}$, $\Psi = \widetilde{\nabla}\ell$, and $\psi(X) = \mathfrak{n}(\widetilde{\nabla}_X \ell)$.*

Eq. (6.8) and Eqs. (6.9)–(6.11) are known as the *Gauss* equation and the *Codazzi* equations of the general hypersurface Σ, respectively. As in the physics literature (*cf.* [19,45,54]), the geometric quantities $\{K, \Psi, \psi\}$ are interpreted as the *intrinsic, extrinsic, and normal second fundamental forms* of Σ, respectively. If metric $\tilde{g}_0$ is Lorentzian with signature $\{-1, +1, \ldots, +1\}$, the rigging field ℓ can be chosen as time-like, whose trajectory thus corresponds to the worldline of an observer. On the other hand, if $\tilde{g}_0|_\Sigma$ is non-degenerate, then ℓ can be chosen as the unit normal vector field, and Eqs. (6.8)–(6.11) reduce to the usual Gauss-Codazzi equations in §2.2.

Proof of Theorem 6.2. The proof consists of three steps. We emphasize the difference between the case of general hypersurfaces and the case of semi-Riemannian submanifolds (Theorem 5.1), while the parallel arguments are only briefly sketched.

1. We first deduce the Gauss–Codazzi equations (6.8)–(6.11) from the immersion ι. As in §2, these equations are obtained by expressing the flatness of $\mathbb{R}^{n+1}$ (that is, the Riemann curvature $\widetilde{R}=0$) with respect to the orthogonal splitting $T_x\mathbb{R}^{n+1} \cong T_x\Sigma \oplus \operatorname{span}(\ell_x)$. Indeed, from the definition of the Riemann curvature, we have

$$\begin{aligned}\widetilde{R}(X,Y)Z =& R(X,Y)Z - K(X,\widetilde{\nabla}_Y Z) - K(X,\widetilde{\nabla}_Y Z)\ell + K(Y,\widetilde{\nabla}_X Z)\ell\\ &- \widetilde{\nabla}_X\big(K(Y,Z)\ell\big) + \widetilde{\nabla}_Y\big(K(X,Z)\ell\big) + K([X,Y],Z)\ell. \end{aligned} \tag{6.12}$$

The detailed computation can be found in [45, §3], with a slightly different sign convention for $\widetilde{R}$. The Gauss equation is obtained by contracting with α. Since $\alpha(\ell)=0$, we have

$$0=\alpha(\widetilde{R}(X,Y),Z)=\alpha(R(X,Y)Z)-\alpha(\widetilde{\nabla}_X(K(Y,Z)\ell))+\alpha(\widetilde{\nabla}_Y(K(X,Z)\ell)),$$

which yields (6.8) by the definition of Ψ.

To obtain the Codazzi equation (6.9), we consider

$$\mathfrak{n}(R(X,Y)Z)=0,$$

where $\mathfrak{n}$ is the normal 1-form. Invoking Eq. (6.12) for $\widetilde{R}$ again yields

$$\begin{aligned}\mathfrak{n}(\widetilde{R}(X,Y)Z) =& -K(X,\nabla_Y Z)+K(Y,\nabla_X Z)-K([X,Y],Z)\\ &-\mathfrak{n}(\widetilde{\nabla}_X(K(Y,Z)\ell))+\mathfrak{n}(\widetilde{\nabla}_Y(K(X,Z)\ell)).\end{aligned}$$

On the other hand, the Leibniz rule of the connection gives us

$$\mathfrak{n}(\widetilde{\nabla}_Y(K(X,Z)\ell)) = YK(X,Z)\mathfrak{n}(\ell)-K(X,Z)\mathfrak{n}(\nabla_Y\ell),$$

which, together with $\mathfrak{n}(\ell)=1$ and the definition of ψ, implies Eq. (6.9).

Next, we consider $\widetilde{R}(X,Y)\ell := \widetilde{\nabla}_X\widetilde{\nabla}_Y\ell - \widetilde{\nabla}_Y\widetilde{\nabla}_X\ell + \widetilde{\nabla}_{[X,Y]}\ell$. Notice that

$$\widetilde{\nabla}_X\widetilde{\nabla}_Y\ell = \sum_{i=1}^n X\Psi(\theta^i,Y)e^i + X\psi(Y)\ell + \sum_{i=1}^n \Psi(\theta^i,Y)\widetilde{\nabla}_X e^i + \psi(Y)\widetilde{\nabla}_X\ell,$$

where the following key identities are utilized:

$$\widetilde{\nabla}_X Y = \nabla_X Y - K(X,Y)\ell, \quad \widetilde{\nabla}_X\ell = \sum_{i=1}^n \Psi(\theta^i,X)e^i + \psi(X)\ell.$$

Then

$$\begin{aligned}\widetilde{\nabla}_X\widetilde{\nabla}_Y\ell = &\sum_{i=1}^n X\Psi(\theta^i,Y)e^i + X\psi(Y)\ell + \sum_{i=1}^n \Psi(\theta^i,Y)\nabla_X e^i\\ &- \sum_{i=1}^n \Psi(\theta^i,Y)K(e^i,X)\ell + \sum_{i=1}^n \psi(Y)\psi(\theta^i,X)e^i + \psi(X)\psi(Y)\ell.\end{aligned}$$

A similar expression holds for $\widetilde{\nabla}_Y\widetilde{\nabla}_X\ell$ by interchanging X and Y:

$$\begin{aligned}\widetilde{\nabla}_Y\widetilde{\nabla}_X\ell = &\sum_{i=1}^n Y\Psi(\theta^i,X)e^i + Y\psi(X)\ell + \sum_{i=1}^n \Psi(\theta^i,X)\nabla_Y e^i\\ &- \sum_{i=1}^n \Psi(\theta^i,X)K(e^i,Y)\ell + \sum_{i=1}^n \psi(X)\psi(\theta^i,Y)e^i + \psi(Y)\psi(X)\ell.\end{aligned}$$

Thus, contracting with $\alpha \in \Omega^1(\Sigma)$ and noting that $\alpha(\ell) = 0$, we conclude the Codazzi equation (6.10).

Finally, the Codazzi equation (6.11) is obtained by contracting $\widetilde{R}(X,Y)\ell$ with the normal 1-form $\mathfrak{n}$. Similarly to the above computations, we have

$$\begin{aligned}\mathfrak{n}(\widetilde{\nabla}_X\widetilde{\nabla}_Y\ell) &= \mathfrak{n}(\widetilde{\nabla}_X(\sum_{i=1}^n \theta^i(\widetilde{\nabla}_Y\ell)e^i + \mathfrak{n}(\nabla_Y\ell)\ell))\\ &= X\psi(Y) + \sum_{i=1}^n \theta^i(\widetilde{\nabla}_Y\ell)\mathfrak{n}(\widetilde{\nabla}_X e^i) + \psi(Y)\psi(X)\\ &= X\psi(Y) + \psi(Y)\psi(X) - \sum_{i=1} K(e^i,X)\Psi(\theta^i,Y),\end{aligned}$$

thanks to another important identity:

$$\mathfrak{n}(\widetilde{\nabla}_X Y) = -\mathfrak{n}(K(Y,X)\ell) = -K(Y,X). \tag{6.13}$$

Therefore, computing for $\mathfrak{n}(\widetilde{\nabla}_Y\widetilde{\nabla}_X\ell)$ in the similar manner:

$$\mathfrak{n}(\widetilde{\nabla}_Y\widetilde{\nabla}_X\ell) = Y\psi(X) + \psi(X)\psi(Y) - \sum_{i=1} K(e^i,Y)\Psi(\theta^i,X),$$

we can deduce Eq. (6.11). Furthermore, observe that the above computations still hold in the sense of distributions for immersions with lower regularity, *i.e.*, $\iota \in W^{2,p}_{\mathrm{loc}}(\Sigma;\mathbb{R}^{n+1})$. This proves the first part of the theorem.

2. Now we tackle the *realization problem*, *i.e.*, finding an immersion ι from Eqs. (6.8)–(6.11). As in the semi-Riemannian submanifolds case, the key is to verify the second structural system (2.13) for a suitable connection 1-form.

For this purpose, we invoke the following identity for differential forms:

$$\mathrm{d}\beta(X,Y) = X\beta(Y) - Y\beta(X) + \beta([X,Y]),$$

where $\beta \in \Gamma(T^*\Sigma)$ is arbitrary. Thus, we can rewrite the three Codazzi equations as

$$\begin{cases} \mathrm{d}K(X,Y,Z) = K(Y,\nabla_X Z) - K(X,\nabla_Y Z) + \psi(X)K(Y,Z) - K(X,Z)\psi(Y), \\ \mathrm{d}\Psi(\alpha,X,Y) = \psi(X)\Psi(\alpha,Y) - \psi(Y)\Psi(\alpha,X) \\ \qquad\qquad + \sum_{i=1}^n \big(\Psi(\theta^i,X)\alpha(\nabla_Y e^i) - \Psi(\theta^i,Y)\alpha(\nabla_X e^i)\big), \\ \mathrm{d}\psi(X,Y) = \sum_{i=1}^n \big(\Psi(\theta^i,Y)K(X,e^i) - \Psi(\theta^i,X)K(Y,e^i)\big). \end{cases} \tag{6.14}$$

Now, define the connection 1-form $\mathcal{W}_\Sigma \in W^{1,p}_{\mathrm{loc}}(\Sigma;\, T^*\Sigma \otimes \mathfrak{gl}(n+1;\mathbb{R}))$ by

$$\mathcal{W}_\Sigma := \begin{bmatrix} \Gamma & \Psi \\ K & \psi \end{bmatrix}. \tag{6.15}$$

More precisely, in the local coordinates, we write

$$\mathcal{W}_\Sigma := \begin{bmatrix} \Gamma^k_{ij}\theta^k & \Psi(\theta^i,\cdot) \\ K(e^i,\cdot) & \psi(\cdot) \end{bmatrix},$$

where, as usual, the Christoffel symbols are defined via $\nabla_{e^i} e^j = \sum_{k=1}^n \Gamma^k_{ij} e^k$ and computed from $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$. The block-matrix representation of $\mathcal{W}$ in Eq. (6.15) is interpreted via the following identifications:

$$\begin{cases} \Gamma = \Gamma^k_{ij}\theta^k \in W^{1,p}_{\mathrm{loc}}(\Sigma\,;\, T^*\Sigma \otimes \mathfrak{gl}(n;\mathbb{R})), \\ \Psi, K \in W^{1,p}_{\mathrm{loc}}(\Sigma\,;\, T^*\Sigma \otimes \mathbb{R}^n), \\ \psi \in W^{1,p}_{\mathrm{loc}}(\Sigma\,;\, T^*\Sigma). \end{cases}$$

Thus, we can recast the Gauss equation (6.8) and the Codazzi equations in the form of (6.14) into the following schematic equalities:

$$\begin{cases} \mathrm{d}K = K\Gamma - \Gamma K - K\psi + \psi K = K \wedge \Gamma - K \wedge \psi, \\ \mathrm{d}\Psi = \Gamma\Psi - \Psi\Gamma + \Psi\psi - \psi\Psi = \Gamma \wedge \Psi - \Psi \wedge \psi, \\ \mathrm{d}\psi = K\Psi - \Psi K = K \wedge \Psi, \end{cases}$$

where the juxtaposition of matrices (*e.g.*, $K\Gamma$) denotes the matrix multiplication, and $\wedge$ is an intertwining of the wedge product on differential forms and the matrix multiplication.

On the other hand, simple manipulations on block matrices lead to

$$\mathcal{W}_\Sigma \wedge \mathcal{W}_\Sigma = \begin{bmatrix} \Gamma \wedge \Gamma + \Psi \wedge K & \Gamma \wedge \Psi + \psi \wedge \Psi \\ K \wedge \Gamma + K \wedge \psi & K \wedge \Psi \end{bmatrix}. \tag{6.16}$$

In this notation, the Riemann curvature is given by

$$R = \mathrm{d}\Gamma - \Gamma \wedge \Gamma \in L^p_{\mathrm{loc}}(\Sigma\,;\, \wedge^2 T^*\Sigma \otimes \mathfrak{gl}(n;\mathbb{R})).$$

Then the preceding two equations yield

$$\mathrm{d}\mathcal{W}_\Sigma - \mathcal{W}_\Sigma \wedge \mathcal{W}_\Sigma = 0, \tag{6.17}$$

i.e., the second structural system as in Eq. (2.13).

Invoking again Lemma 5.1, we obtain the local solution $A \in W^{1,p}_{\mathrm{loc}}(U \subset \Sigma; \mathfrak{gl}(n+1, \mathbb{R}))$ to the following Pfaff system:

$$\mathrm{d}A = \mathcal{W}_\Sigma \cdot A,$$

where $U \subset \Sigma$ is an open trivialized neighborhood.

3. Now we solve the local isometric immersion $\iota : \Sigma \hookrightarrow \mathbb{R}^{n+1}$ via the Poincaré system:

$$\mathrm{d}\iota = \tilde{\theta} \cdot A,$$

where $\tilde{\theta} = (\theta^1, \ldots, \theta^n, 0)^\top : U \subset \Sigma \to \mathbb{R}^{n+1} \otimes T^*\Sigma$ is the $\mathbb{R}^{n+1}$-valued differential 1-form. As before, it is solvable if and only if the following first structural system is satisfied:

$$\mathrm{d}\tilde{\theta} = \tilde{\theta} \wedge \mathcal{W}_\Sigma.$$

Recall that the first structural system holds whenever the affine connection $\nabla = \iota^* \widetilde{\nabla}$ is torsion-free (see Appendix A.5). Here, as $K(X,Y) = K(Y,X)$ (*cf.* [40,45]), the torsion-free condition is verified, which leads to the existence of a solution $\iota \in W^{2,p}_{\mathrm{loc}}(U; \mathbb{R}^{n+1})$.

The assertion now follows from the proof of Theorem 5.1. This completes the proof.

Remark 6.1 Theorem 6.2 was proved locally in [41] by computations in the local coordinates. Our proof above, being global and intrinsic in nature, both helps clarify the geometric meanings of $\{K, \Psi, \psi\}$ and serves as a crucial step towards the establishment of the weak rigidity theorem, Theorem 6.3, for general hypersurfaces below.

In the proof above, it is crucial to establish the equivalence of the Gauss–Codazzi equations (6.8)–(6.11) with Eq. (6.17), namely the second structural system for $\mathcal{W}_\Sigma$, which is defined in Eq. (6.15) in terms of the Christoffel symbol Γ and the intrinsic, extrinsic, and normal second fundamental forms $\{K, \Psi, \psi\}$. Therefore, by invoking the quadratic theorem (Theorem 3.1) and establishing the weak continuity of $\mathrm{d}\mathcal{W}_\Sigma = \mathcal{W}_\Sigma \wedge \mathcal{W}_\Sigma$ again, we arrive at the weak rigidity theorem for the general hypersurfaces:

Theorem 6.3 *Let (Σ, g) be a simply-connected n-dimensional hypersurface of semi-Euclidean space $\mathbb{R}^{n+1}$ with $\mathrm{Ind}(\Sigma) = \nu$ and $g \in W^{1,p}_{\mathrm{loc}}(\Sigma, O(\nu, n-\nu))$ for $p > n$. Let $\{f_\varepsilon\}$ be a family of immersions of semi-Riemannian submanifolds uniformly bounded in $W^{2,p}_{\mathrm{loc}}(\Sigma; \mathbb{R}^{n+k})$, and let $\{l_\varepsilon\}$ be an associated family of rigging vector fields uniformly bounded in $W^{1,p}_{\mathrm{loc}}(\Sigma; T\Sigma)$. Denote by $\{K_\varepsilon, \Psi_\varepsilon, \psi_\varepsilon\}$ the corresponding intrinsic, extrinsic, and normal second fundamental forms. Then, after passing to a subsequence if necessary, $\{f_\varepsilon\}$ converges to an immersion $f \in W^{2,p}_{\mathrm{loc}}(\Sigma; \mathbb{R}^{n+1})$ in the sense of distributions; in addition, its intrinsic, extrinsic, and normal second fundamental forms are weak limits in L^p_{loc} of $\{K_\varepsilon\}$, $\{\Psi_\varepsilon\}$, and $\{\psi_\varepsilon\}$, respectively.*

Proof First, thanks to Eq. (6.16), all the entries of the 2–form-valued matrix $\mathcal{W}^\Sigma \wedge \mathcal{W}^\Sigma$ are linear combinations of the quadratic forms in $\mathit{\Gamma}, \mathit{\Psi}, K$, and ψ, each of which lies in $W^{1,p}_{\rm loc}$. Thus, $\mathcal{W}^\Sigma \wedge \mathcal{W}^\Sigma \in W^{1,\frac{p}{2}}_{\rm loc}$ by the Cauchy–Schwarz inequality, which is compactly embedded in $W^{-1,q}_{\rm loc}$ for some $1 < q < 2$, as computed in Step 3 of the proof of Theorem 5.2.

On the other hand, $\mathcal{W}^\Sigma \in W^{1,p}_{\rm loc}$ implies that $\mathrm{d}\mathcal{W}^\Sigma \in L^p_{\rm loc}$, which is compactly embedded into $W^{-1,p}_{\rm loc}$ by the Rellich lemma. Using Eq. (6.16) and the interpolations of Sobolev spaces, we deduce that $\{\mathrm{d}\mathcal{W}^\Sigma_\varepsilon\}$ is pre-compact in $H^{-1}_{\rm loc}$.

Therefore, with the above pre-compactness result, the proof proceeds as that for Theorem 5.2. In particular, we establish the weak continuity of the Cartan structural system $\mathrm{d}\mathcal{W}^\Sigma = \mathcal{W}^\Sigma \wedge \mathcal{W}^\Sigma$. Then, in view of the realization theorem (Theorem 6.2) for general hypersurfaces, it implies the existence of the limiting immersion f, together with a rigging vector field ℓ, for which the intrinsic, extrinsic, and normal second fundamental forms $\{K, \mathit{\Psi}, \psi\}$ are well-defined. After passing to a subsequence if necessary, $\{K_\varepsilon, \mathit{\Psi}_\varepsilon, \psi_\varepsilon\}$ converges in the weak $L^p_{\rm loc}$ topology to $\{K, \mathit{\Psi}, \psi\}$ due to the uniqueness of weak limits. Then the proof is completed.

Appendix A Proofs of Several Semi-Riemannian Geometric Theorems

In this appendix, we provide the proofs of several semi-Riemannian geometric theorems, whose Riemannian analogues are well-known. These results are viewed as *folklores* in the geometric community, but the proofs appear elusive in the literature. For the convenience of the reader, now we carefully write down the complete proofs in detail below.

A.1 **Proof of Theorem 2.1**

The derivation of Eqs. (2.8)–(2.9) can be found on page 100 and page 115 in [51], respectively. It remains to derive the Ricci equation (2.10). Indeed, we have

$$\begin{aligned}
0 &= \widetilde{R}(X, Y, \xi, \eta) \\
&= \langle \widetilde{\nabla}_X \widetilde{\nabla}_Y \xi, \eta\rangle - \langle \widetilde{\nabla}_Y \widetilde{\nabla}_X \xi, \eta\rangle + \langle \widetilde{\nabla}_{[X,Y]}\xi, \eta\rangle \\
&= \langle \nabla^\perp_X \nabla^\perp_Y \xi, \eta\rangle - \langle \nabla^\perp_X (S_\xi Y), \eta\rangle - \langle \nabla^\perp_Y \nabla^\perp_X \xi, \eta\rangle + \langle \nabla^\perp_Y (S_\xi X), \eta\rangle + \langle \nabla^\perp_{[X,Y]}\xi, \eta\rangle \\
&= R^\perp(X, Y, \xi, \eta) - \langle \nabla^\perp_X (S_\xi Y), \eta\rangle + \langle \nabla^\perp_Y (S_\xi X), \eta\rangle,
\end{aligned}$$

in view of the definition for $R^\perp$. Moreover, owing to the self-adjointness of S_η, we have

$$\langle \nabla^\perp_X (S_\xi Y), \eta\rangle = X\langle S_\xi Y, \eta\rangle - \langle S_\xi Y, \tan(\widetilde{\nabla}_X \eta)\rangle = \langle S_\xi Y, S_\eta X\rangle = \langle S_\eta \circ S_\xi(Y), X\rangle,$$

and similarly $\langle \nabla^\perp_Y (S_\xi X), \eta\rangle = \langle S_\xi \circ S_\eta(Y), X\rangle$. Then Eq. (2.10) follows.

A.2 Proof of Lemma 2.1

The connection 1-form $\mathcal{W}$ (Definition 2.8) is semi-skew-symmetric. It is crucial to observe that a matrix $B \in O(\nu, n-\nu)$ if and only if its transpose $B^\top$ takes the form:

$$B^\top = \epsilon_{n,\nu} B^{-1} \epsilon_{n,\nu}. \tag{A.1}$$

The signature matrix $\epsilon_{n,\nu}$ is defined in Eq. (2.6).

We first observe that, for each $Z \in O(\nu, n-\nu)$,

$$T_Z O(\nu, n-\nu) = \Big\{ A \in \mathfrak{gl}(n;\mathbb{R}) \,:\, Z\epsilon_{n,\nu} A^\top + A\epsilon_{n,\nu} Z^\top = 0 \Big\}. \tag{A.2}$$

Indeed, let $\sigma : (-\varepsilon, \varepsilon) \to O(\nu, n-\nu)$ be a C^1-curve with $\sigma(0) = Z$. In view of Eq. (A.1), we have

$$\sigma(t)^\top = \epsilon_{n,\nu} \sigma(t)^{-1} \epsilon_{n,\nu}.$$

Taking the derivative in t yields

$$\dot{\sigma}(t)^\top = -\epsilon_{n,\nu} \sigma(t)^{-1} \dot{\sigma}(t) \sigma(t)^{-1} \epsilon_{n,\nu}.$$

Thus, evaluating the above at $t = 0$ and using the identity: $Z^\top = \epsilon_{n,\nu} Z^{-1} \epsilon_{n,\nu}$, we have

$$\dot{\sigma}(0)^\top = -Z^\top \epsilon_{n,\nu} \dot{\sigma}(0) \epsilon_{n,\nu} Z^\top.$$

Since the elements of $T_Z O(\nu, n-\nu)$ are in one-to-one correspondence with $\dot{\sigma}(0)$ for such σ, Eq. (A.2) follows. As a consequence,

$$\mathfrak{o}(\nu, n-\nu) := T_{\mathrm{Id}} O(\nu, n-\nu) = \Big\{ A \in \mathfrak{gl}(n;\mathbb{R}) \,:\, \epsilon_{n,\nu} A^\top + A\epsilon_{n,\nu} = 0 \Big\}.$$

We now verify that $\mathcal{W}$ lies in the Lie algebra of the semi-orthogonal group. Clearly, it suffices to prove that, for each $a, b \in \{1, \ldots, n+k\}$,

$$\epsilon^a \omega_a^b = -\epsilon^b \omega_b^a.$$

Indeed, for $a = i$ and $b = \alpha$, this follows by the definition of ω_i^α in the fourth equation of (2.11). For $a = i$ and $b = j$, as ∇ is compatible with metric g, we deduce from the first equation of (2.11) that

$$0 = \partial_l \langle \partial_i, \partial_j \rangle = \langle \nabla_{\partial_l} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_l} \partial_j \rangle = \epsilon^j \omega_j^i(\partial_l) + \epsilon^i \omega_i^j(\partial_l).$$

Finally, for $a = \alpha$ and $b = \beta$, it follows from a similar computation by using the third equation of (2.11), thanks to the compatibility of ∇^E with g^E. This completes the proof.

A.3 Proof of Proposition 2.1

We divide the arguments into four steps.

1. We begin by observing that the definition of the connection 1-form $\mathcal{W}$, *i.e.*, Eq. (2.11), implies that

$$\nabla_{\partial_i}\partial_j = \sum_l \omega_j^l(\partial_i)\partial_l, \quad \mathrm{II}(\partial_i,\partial_j) = \sum_\alpha \omega_\alpha^i(\partial_j)\partial_\alpha, \quad \nabla^E_{\partial_i}\partial_\alpha = \sum_\beta \omega_\beta^\alpha(\partial_i)\partial_\beta.$$

One may deduce the following identities of the shape operator S:

$$S_{\partial_i}\partial_\alpha = \sum_j \epsilon^j\langle S_{\partial_i}\partial_\alpha, \partial_j\rangle\partial_j = \sum_j \epsilon^j\langle \mathrm{II}(\partial_i,\partial_j),\partial_\alpha\rangle\partial_j = \sum_j \epsilon^j\epsilon^\alpha\omega_\alpha^i(\partial_j)\partial_j.$$

2. Next, the Gauss equation (2.8) is equivalent to

$$R(\partial_i,\partial_j,\partial_k) = S_{\partial_i}\mathrm{II}(\partial_j,\partial_k) - S_{\partial_j}\mathrm{II}(\partial_i,\partial_k). \tag{A.3}$$

Applying the symmetry of II twice (in the first and third equalities below), we obtain

$$\begin{aligned}
S_{\partial_i}\mathrm{II}(\partial_j,\partial_k) - S_{\partial_j}\mathrm{II}(\partial_i,\partial_k) &= S_{\partial_i}\mathrm{II}(\partial_k,\partial_j) - S_{\partial_j}\mathrm{II}(\partial_k,\partial_i)\\
&= S_{\partial_i}\big(\sum_\alpha \omega_\alpha^k(\partial_j)\partial_\alpha\big) - S_{\partial_j}\big(\sum_\alpha \omega_\alpha^k(\partial_i)\partial_\alpha\big)\\
&= \sum_\alpha\sum_l \epsilon^\alpha\epsilon^l\big(\omega_\alpha^k(\partial_j)\omega_\alpha^l(\partial_i) - \omega_\alpha^k(\partial_i)\omega_\alpha^l(\partial_j)\big)\partial_l\\
&= \sum_\alpha\sum_l (\omega_\alpha^k \wedge \omega_l^\alpha)(\partial_i,\partial_j)\partial_l,
\end{aligned}$$

where the last equality follows from Lemma 2.1. On the other hand, the Riemann curvature of the Levi–Civita connection on TM is computed directly from the definition:

$$\begin{aligned}
R(\partial_i,\partial_j,\partial_k) &:= \nabla_i\nabla_j\partial_k - \nabla_j\nabla_i\partial_k + \nabla_{[\partial_i,\partial_j]}\partial_k\\
&= \sum_l \big\{\partial_i(\omega_k^l(\partial_j))\partial_l - \partial_j(\omega_k^l(\partial_i))\partial_l + \omega_k^l([\partial_i,\partial_j])\partial_l\\
&\qquad + \omega_k^l(\partial_j)\sum_s \omega_l^s(\partial_i)\partial_s - \omega_k^l(\partial_i)\sum_s \omega_l^s(\partial_j)\partial_s\big\}\\
&= \sum_s \big\{\mathrm{d}\omega_k^s - \sum_l \omega_k^l\wedge\omega_l^s\big\}(\partial_i,\partial_j)\partial_s.
\end{aligned}$$

Equating the preceding computations via Eq. (A.3), we conclude that

$$\mathrm{d}\omega_k^s = \sum_b \omega_b^k \wedge \omega_s^b.$$

3. Applying the same argument to $R^E(\partial_i,\partial_j,\partial_\gamma)$ and utilizing the Ricci equation (2.10), we deduce that $\mathrm{d}\omega_\beta^\alpha = \sum_b \omega_b^\alpha \wedge \omega_\beta^b$.

Furthermore, starting with the Codazzi equations (2.9), we have

$$\begin{aligned}
0 &= \widetilde{\nabla}_{\partial_i}\mathrm{II}(\partial_j,\partial_k) - \widetilde{\nabla}_{\partial_j}\mathrm{II}(\partial_i,\partial_k)\\
&= \sum_\gamma \partial_i(\omega^j_\gamma(\partial_k))\partial_\gamma + \sum_\beta \omega^j_\beta(\partial_k)\sum_\gamma \omega^\beta_\gamma(\partial_i)\partial_\gamma - \sum_\gamma \partial_j(\omega^i_\gamma(\partial_k))\partial_\gamma\\
&\quad - \sum_\beta \omega^i_\beta(\partial_k)\sum_\gamma \omega^\beta_\gamma(\partial_j)\partial_\gamma\\
&= \sum_\gamma \Big\{\partial_i(\omega^k_\gamma(\partial_j)) - \partial_j(\omega^k_\gamma(\partial_i)) - \sum_\beta \big(\omega^k_\beta(\partial_i)\omega^\beta_\gamma(\partial_j) - \omega^k_\beta(\partial_j)\omega^\beta_\gamma(\partial_i)\big)\Big\}\partial_\gamma\\
&= \sum_\gamma \Big\{\mathrm{d}\omega^k_\gamma(\partial_i,\partial_j) - \omega^k_\gamma[\partial_i,\partial_j] - \sum_\beta(\omega^k_\beta\wedge\omega^\beta_\gamma)(\partial_i,\partial_j)\Big\}\partial_\gamma. \qquad \text{(A.4)}
\end{aligned}$$

In the penultimate equality, we have used the self-adjointness of II, *i.e.*, $\omega^i_\alpha(\partial_j) = \omega^j_\alpha(\partial_i)$. The final equality follows from the definition of $\mathrm{d}\omega^k_\gamma$ and $\omega^k_\beta\wedge\omega^\beta_\gamma$.

To compute the Lie bracket term in the last equality of Eq. (A.4), we invoke the torsion-free condition of the affine connection:

$$\begin{aligned}
\sum_\gamma \omega^k_\gamma[\partial_i,\partial_j]\partial_\gamma &= \sum_\gamma \omega^k_\gamma\big(\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i\big)\partial_\gamma\\
&= \sum_\gamma\sum_l \omega^k_\gamma\big(\omega^l_j(\partial_i)\partial_l - \omega^l_i(\partial_j)\partial_l\big)\partial_\gamma\\
&= \sum_\gamma\sum_l \big(\omega^l_j(\partial_i)\omega^l_\gamma(\partial_k) - \omega^l_i(\partial_j)\omega^l_\gamma(\partial_k)\big)\partial_\gamma\\
&= \sum_\gamma\sum_l (\omega^k_l\wedge\omega^l_\gamma)(\partial_i,\partial_j)\partial_\gamma,
\end{aligned}$$

again owing to the symmetries of the second fundamental form and $\mathcal{W}\wedge\mathcal{W}$. Substituting it back to Eq. (A.4) yields that $\mathrm{d}\omega^k_\gamma = \sum_b \omega^k_b\wedge\omega^b_\gamma$.

4. Combining Steps 1–3 together, we conclude

$$\mathrm{d}\omega^a_c = \sum_b \omega^a_b\wedge\omega^b_c.$$

Moreover, as an equation on $\Omega^2\big(\mathfrak{o}(\nu+\tau,(n+k)-(\nu+\tau))\big)$, Eq. (2.13) is independent of the choice of moving frames. This completes the proof.

A.4 **Derivation of the First Structural System** (2.15)

We now present a derivation of the first structural system (2.15) and show that it is equivalent to the torsion-free condition of the affine connection.

We compute the Lie bracket of the basic vector fields ∂_i and ∂_j in two different ways. On one hand, we have

$$[\partial_i,\partial_j] = \sum_l \epsilon^l\theta^l[\partial_i,\partial_j]\partial_l.$$

On the other hand, the torsion-free condition of ∇ gives

$$\begin{aligned}[\partial_i,\partial_j] &= \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i \\ &= \nabla_{\partial_i}\big(\epsilon^j\sum_k \theta^k(\partial_j)\partial_k\big) - \nabla_{\partial_j}\big(\epsilon^i\sum_k \theta^k(\partial_i)\partial_k\big) \\ &= \sum_k \big(\epsilon^k\delta_j^k\nabla_{\partial_i}\partial_k - \epsilon^k\delta_i^k\nabla_{\partial_j}\partial_k\big) \\ &= \sum_l \epsilon^l\big(\omega_l^i(\partial_j) - \omega_l^j(\partial_i)\big)\partial_l,\end{aligned}$$

where the last equality follows from the *semi-skew-symmetry* of the connection 1-form $\mathcal{W}$ (Lemma 2.1). Then

$$\theta^l[\partial_i,\partial_j] = \omega_l^i(\partial_j) - \omega_l^j(\partial_i). \tag{A.5}$$

Now we observe

$$\mathrm{d}\theta^l(\partial_i,\partial_j) = \partial_i(\theta^l(\partial_j)) - \partial_j(\theta^l(\partial_i)) + \theta^l[\partial_i,\partial_j] = \theta^l[\partial_i,\partial_j],$$

and

$$\begin{aligned}\sum_k(\theta^k\wedge\omega_k^l)(\partial_i,\partial_j) &= \sum_k\big(\theta^k(\partial_i)\omega_k^l(\partial_j) - \theta^k(\partial_j)\omega_k^l(\partial_i)\big) \\ &= \sum_k \delta_i^k\omega_k^l(\partial_j) - \delta_j^k\omega_k^l(\partial_i) = \omega_i^l(\partial_j) - \omega_j^l(\partial_i).\end{aligned}$$

Utilizing Eq. (A.5), we obtain

$$\mathrm{d}\theta^l = \sum_k \theta^k\wedge\omega_k^l.$$

As Eq. (2.15) is independent of the choice of local coordinates, This completes the proof.

A.5 Proof of Theorem 5.1 in the C^∞ Case

We now present a proof of the realization theorem in the C^∞ case, following Tenenblat's arguments in [61] for the Riemannian case. We emphasize that various modifications are necessary due to the semi-Riemannian geometry. We divide the arguments into four steps.

1. We start with solving a *Pfaff system* for the bundle connection A on $TM\oplus E$. More precisely, we show that, for any $x_0\in M$, the following initial value problem for first-order PDEs:

$$\mathcal{W} = \mathrm{d}A\cdot A^{-1}, \qquad A(0) = A(x_0)\in O(\nu+\tau,(n+k)-(\nu+\tau)), \tag{A.6}$$

has a solution $A\in C^\infty(U;O(\nu+\tau,(n+k)-(\nu+\tau)))$ in some neighborhood U of x_0.

Indeed, without loss of generality, assume that $x_0 = 0$ in the local coordinate $\{\partial_i\}_1^n$. Also, take $Z = \{Z^a_b\}$ as the canonical frame field on $\mathfrak{gl}(n+k;\mathbb{R}) \cong \mathbb{R}^{(n+k)^2}$, with signature inherited from $\widetilde{M} = \mathbb{R}^{n+k}_{\nu+\tau}$. For example, $Z := \tilde{g}_0$ is one suitable choice. Motivated by [61], we consider the following map:

$$\begin{aligned}\Lambda^{(x,Z)} : T_xM \times T_ZO(\nu+\tau,(n+k)-(\nu+\tau)) &\longrightarrow T_ZO(\nu+\tau,(n+k)-(\nu+\tau)),\\ (X,\mathfrak{m}) \quad &\longmapsto \quad \mathrm{d}_xZ(\mathfrak{m}) + Z\cdot\mathcal{W}(X)|_x,\end{aligned}$$

which is abbreviated in the sequel as

$$\Lambda^{(x,Z)} = \mathrm{d}Z - \mathcal{W}\cdot Z. \tag{A.7}$$

Using the characterization of tangent spaces of the semi-orthogonal group and its Lie algebra (*cf.* the proof of Lemma 2.1), we see that $\Lambda^{(x,Z)}$ is well-defined. Indeed,

$$\begin{aligned}&Z\tilde{g}_0(\Lambda^{(x,Z)}(X,\mathfrak{m}))^\top + \Lambda^{(x,Z)}(X,\mathfrak{m})\tilde{g}_0Z^\top\\ &= Z\tilde{g}_0(\mathrm{d}Z(\mathfrak{m})^\top - Z^\top\mathcal{W}(X)^\top) + \big(\mathrm{d}Z(\mathfrak{m}) - \mathcal{W}(X)Z\big)\tilde{g}_0Z^\top\\ &= -\big(\tilde{g}_0\mathcal{W}(X)^\top + \mathcal{W}(X)\tilde{g}_0\big) + \big(Z\tilde{g}_0\mathrm{d}Z(\mathfrak{m})^\top + \mathrm{d}Z(\mathfrak{m})\tilde{g}_0Z^\top\big)\\ &= 0,\end{aligned}$$

since

$$\mathrm{d}Z(\mathfrak{m}) \in T_ZO(\nu+\tau,(n+k)-(\nu+\tau)), \quad \mathcal{W}(X) \in \mathfrak{o}(\nu+\tau,(n+k)-(\nu+\tau)).$$

Next, we define the following *distribution* in the Frobenius sense:

$$\mathcal{D}^{(x,Z)} := \ker(\Lambda^{(x,Z)}) \subset T_xM \times T_ZO(\nu+\tau,(n+k)-(\nu+\tau)).$$

Our goal is to show that it is *completely integrable*. Assuming so, we can find the unique maximal integral submanifold in some neighborhood of x_0. Notice that

$$\mathcal{D}^{(0,Z)} \cap \big(\{0\} \oplus T_ZO(\nu+\tau,(n+k)-(\nu+\tau))\big) = \{0\},$$

i.e., the distribution is transverse to the TM factor at point $x_0 = 0$, because

$$\Lambda^{(x,Z)}(0,\mathfrak{m}) = \mathfrak{m}.$$

In view of the classical implicit function theorem, $\mathcal{D}^{(x,Z)}$ is locally a graph of a smooth function A from TU to $TO(\nu+\tau,(n+k)-(\nu+\tau))$, with x lies in U, an open neighborhood of x_0. This function A solves the Pfaff system (A.6) in view of the definition of $\Lambda^{(x,Z)}$.

2. It now remains to prove the complete integrability of distribution $\mathcal{D}^{(x,Z)}$. By the *Frobenius theorem*, we show that $\mathcal{D}^{(x,Z)}$ is *involutive*. That is, for any $(X_i,\mathfrak{m}_i) \in \mathcal{D}^{(x,Z)}$ for $i = 1,2$, the commutator stays in $\mathcal{D}^{(x,Z)}$:

$$\Lambda^{(x,Z)}[(X_1,\mathfrak{m}_1),(X_2,\mathfrak{m}_2)] = 0.$$

Indeed, utilizing the following identity for the exterior differential:

$$\mathrm{d}\Lambda^{(x,Z)}(s_1,s_2)=s_1(\Lambda^{(x,Z)}s_2)-s_2(\Lambda^{(x,Z)}s_1)-\Lambda^{(x,Z)}[s_1,s_2]$$

for $s_1,s_2\in T_xU\oplus T_ZO(\nu+\tau,(n+k)-(\nu+\tau))$, we reduce the problem to proving the identity:

$$\mathrm{d}\Lambda^{(x,Z)}((X_1,\mathfrak{m}_1),(X_2,\mathfrak{m}_2))=0. \tag{A.8}$$

To this end, we compute $\mathrm{d}\Lambda^{(x,Z)}$. Since

$$\Lambda=\mathrm{d}Z-\mathcal{W}\cdot Z,$$

we have

$$\begin{aligned}\mathrm{d}\Lambda&=\mathrm{d}(\mathrm{d}Z)-\mathrm{d}\mathcal{W}\cdot Z+\mathcal{W}\wedge(\mathrm{d}Z)\\&=-\mathcal{W}\wedge\mathcal{W}\cdot Z+\mathcal{W}\wedge(\Lambda+\mathcal{W}\cdot Z)=\mathcal{W}\wedge\Lambda,\end{aligned}$$

where we have used the second structural system (2.13), together with the definition of Λ in Eq. (A.7), for the second equality. As $(X_i,\mathfrak{m}_i)\in\mathcal{D}^{(x,Z)}$ for $i=1,2$, we then have

$$\begin{aligned}&\mathrm{d}\Lambda^{(x,Z)}((X_1,\mathfrak{m}_1),(X_2,\mathfrak{m}_2))\\&=(\mathcal{W}|_x\wedge\Lambda^{(x,Z)})((X_1,\mathfrak{m}_1),(X_2,\mathfrak{m}_2))\\&=\mathcal{W}(X_1,\mathfrak{m}_1)|_x\Lambda^{(x,Z)}(X_2,\mathfrak{m}_2)-\mathcal{W}(X_2,\mathfrak{m}_2)|_x\,\Lambda^{(x,Z)}(X_1,\mathfrak{m}_1)\\&=0.\end{aligned}$$

This completes the proof of Eq. (A.8), which implies that the Pfaff system (A.6) is solvable.

3. Now, define

$$\underset{\sim}{\Theta}=(\theta^1,\dots,\theta^n,0,\dots,0)^\top\in\Omega^1(\mathbb{R}^{n+k})$$

and, for $x_0\in M$, consider the *Poincaré* system:

$$\mathrm{d}f=\underset{\sim}{\Theta}\cdot A,\qquad f(x_0)=f_0, \tag{A.9}$$

where $f_0\in C^\infty(M;\widetilde{M})$ and $\mathrm{d}_{x_0}f_0(v)\neq 0$ for $v\in T_{x_0}M\setminus\{0\}$. Suppose that this system is solvable. Then, as A takes values in $O(\nu+\tau,(n+k)-(\nu+\tau))$, $\det(A)=\pm1$, by using Eq. (A.1). In particular, A is invertible. It follows from Eq. (A.9) that the linear map $\mathrm{d}f$ has rank n, so that f is an immersion indeed.

Solving for f from Eq. (A.9) is equivalent to showing that $\underset{\sim}{\Theta}\cdot A$ is an exact 1-form. For simply-connected M, by the *Poincaré lemma*, it suffices to verify that $\mathrm{d}(\underset{\sim}{\Theta}\cdot A)=0$; that is, it is a closed 1-form. Indeed,

$$\mathrm{d}(\underset{\sim}{\Theta}\cdot A)=\mathrm{d}\underset{\sim}{\Theta}\cdot A-\underset{\sim}{\Theta}\wedge\mathrm{d}A,$$

and we can compute the second term by $\underset{\sim}{\Theta}\wedge\mathrm{d}A=(\underset{\sim}{\Theta}\wedge\mathcal{W})\cdot A$, thanks to the Pfaff system (A.6) solved in Steps 1–2 above. Thus, the exactness of $\underset{\sim}{\Theta}\cdot A$ follows directly from

$$\mathrm{d}\underset{\sim}{\Theta}=\underset{\sim}{\Theta}\wedge\mathcal{W},$$

which is just the first structural system (2.15). Thus, we have established the solvability of the initial value problem for the Poincaré system (A.9).

4. With the immersion f from the Poincaré system, we now identify the normal bundle $TM^\perp := f^*T\widetilde{M}/TM$ with the given bundle E, and identify the second fundamental form induced by f with the given symmetric tensor field II. Moreover, we can deduce the uniqueness of the local immersion up to the rigid motions of $\mathbb{R}^{n+k}_{\nu+\tau}$, *i.e.*, modulo the actions by the semi-Riemannian congruence group $\mathbb{R}^{n+k} \rtimes O(\nu+\tau,(n+k)-(\nu+\tau))$.

4(a). First of all, define an orthonormal frame $\{\widetilde{\partial}_a\}_1^{n+k}$ on $T\widetilde{M}$ via maps f and A solved by the Pfaff and Poincaré systems. We denote by $\{\frac{\partial}{\partial Z^a}\}_1^{n+k}$ the canonical orthonormal basis on $\widetilde{M} = \mathbb{R}^{n+k}_{\nu+\tau}$ with respect to $\tilde{g}_0 = \epsilon_{n\,\nu} \oplus \epsilon_{k\,\tau}$. In this basis, we set

$$\widetilde{\partial}_i := \mathrm{d}f(\partial_i), \qquad \widetilde{\partial}_\alpha := \sum_b A^\alpha_b \frac{\partial}{\partial Z^b}, \tag{A.10}$$

where the definition of $\{\widetilde{\partial}_a\}$ is independent of the choice of bases on $\widetilde{M}$: Recall that, for each $x \in M$, $A(x)$ lies in $O(\nu+\tau,(n+k)-(\nu+\tau)) \subset \mathfrak{gl}(n+k;\mathbb{R}) \cong \mathrm{End}\,T\widetilde{M}$, the group of linear transformations on $T\widetilde{M}$. Using a further identification: $\mathrm{End}\,T\widetilde{M} \cong T^*\widetilde{M} \otimes T\widetilde{M}$, we view $A(x)$ as a linear map from $T\widetilde{M}$ to itself. From this perspective, $\widetilde{\partial}_\alpha$ coincides with A^α, *i.e.*, the normal component of A as a $T\widetilde{M}$-valued function defined on $M \times T\widetilde{M}$. Thus, Eq. (A.10) is equivalent to

$$\widetilde{\partial} = (\mathrm{d}f)^\sharp \oplus \mathrm{nor}A, \tag{A.11}$$

where $\sharp : T^*\widetilde{M} \to T\widetilde{M}$ is the canonical bundle isomorphism turning a 1-form into the corresponding vector field. This gives us an intrinsic definition of frame $\{\widetilde{\partial}_a\}$.

Now we verify that $\{\widetilde{\partial}_a\}$ is indeed an orthonormal frame. First, using the Poincaré system (A.9) defining f, together with the characterization of the semi-orthogonal group (*cf.* Eq. (A.1)), we have

$$\begin{aligned}
\tilde{g}_0(\widetilde{\partial}_i,\widetilde{\partial}_j) &= \tilde{g}_0(\mathrm{d}f(\partial_i),\mathrm{d}f(\partial_j)) = \tilde{g}_0(\underset{\sim}{\Theta}(\partial_i)A,\underset{\sim}{\Theta}(\partial_j)A)\\
&= \underset{\sim}{\Theta}(\partial_i)A^\top \tilde{g}_0 A \underset{\sim}{\Theta}(\partial_j)^\top = \underset{\sim}{\Theta}(\partial_i)\tilde{g}_0\underset{\sim}{\Theta}(\partial_j)^\top\\
&= (\epsilon_{n,\nu})^i_j := \epsilon^i\delta^i_j.
\end{aligned}$$

Also, for the normal directions, using the shorthand notations in Eq. (A.11), we have

$$\tilde{g}_0(\widetilde{\partial}_\alpha,\widetilde{\partial}_\beta) = (\tilde{g}_0(\mathrm{nor}A,\mathrm{nor}A))^\alpha_\beta = \mathrm{nor}(A^\top\tilde{g}_0A)^\alpha_\beta = (\epsilon_{k,\tau})^\alpha_\beta = \epsilon^\alpha\delta^\alpha_\beta.$$

Finally, it follows from the Poincaré system (A.9) that

$$\tilde{g}_0(\partial_i,\partial_\alpha) = \tilde{g}_0(\underset{\sim}{\Theta}\cdot A(\partial_i),A^\alpha) = (\underset{\sim}{\Theta}(\partial_i))^\top(A^\top\tilde{g}_0A)^\alpha = (\tilde{g}_0)^\alpha_i \equiv 0,$$

since $\tilde{g}_0$ is a diagonal matrix. The orthonormality of $\{\widetilde{\partial}_a\}$ now follows.

4(b). Next, we identify the normal bundle induced by f, written as $TM^\perp := f^*T\widetilde{M}/TM$ (Convention 2.2), with the prescribed vector bundle E. For this purpose, we define the following *identification map* on a trivialized chart $U \subset M$:

$$\begin{aligned}\mathcal{I} : (U \subset M) \times (F \cong \mathbb{R}^k) &\longrightarrow \widetilde{M},\\ (x, \xi) &\longmapsto f(x) + \sum_\beta \xi_\beta \widetilde{\partial}_\beta.\end{aligned}$$

Indeed, $\mathrm{d}\mathcal{I} : TU \oplus TF \to T\widetilde{M}$ coincides with $\mathrm{d}f + \mathrm{nor}A$; equivalently, one can write $\mathrm{d}\mathcal{I}(\partial_a) = \widetilde{\partial}_a$ for each $a \in \{1, \ldots, n+k\}$. In particular, $\mathrm{d}\mathcal{I}(TU) \subset TM$ and $\mathrm{d}\mathcal{I}(F) \subset TM^\perp$, which indicates that the identification map $\mathcal{I}$ preserves the horizontal and vertical subspaces of the vector bundles $TM \oplus E$ and $T\widetilde{M}$. Moreover, as f is an immersion (justified in Step 3 above), we deduce that $\mathcal{I}$ is a diffeomorphism, by shrinking chart U if necessary. Thus, we have obtained an identification of E with $TM^\perp$ in the trivialized local charts.

In addition, by the construction of the moving frame $\{\widetilde{\partial}_a\}$ on $T\widetilde{M}$ in Eq. (A.10), we have

$$\begin{aligned}&\mathcal{I}^*\tilde{g}_0\Big(\sum_i U^i\partial_i + \sum_\alpha U^\alpha \partial_\alpha, \sum_j V^j\partial_j + \sum_\beta V^\beta \partial_\beta\Big)\\ &= \tilde{g}_0\Big(\sum_i U^i \mathrm{d}f(\partial_i), \sum_j V^j \mathrm{d}f(\partial_j)\Big) + \tilde{g}_0\Big(\sum_\alpha U^\alpha \widetilde{\partial}_\alpha, \sum_\beta V^\beta \widetilde{\partial}_\beta\Big)\\ &= \sum_i\sum_j U^i V^j \tilde{g}_0(\widetilde{\partial}_i, \widetilde{\partial}_j) + \sum_\alpha\sum_\beta U^\alpha V^\beta \tilde{g}_0(\widetilde{\partial}_\alpha, \widetilde{\partial}_\beta)\\ &= g(U|_{TM}, V|_{TM}) + g^E(U|_E + V|_E)\end{aligned}$$

for any $U, V \in \Gamma(T\widetilde{M})$. It follows that $\mathcal{I}$ is an isometry between $TU \oplus E$ and $T\widetilde{M}$:

$$\mathcal{I}^*\tilde{g}_0 = g \oplus g^E$$

as the block direct sum of matrices. Thus, f is a local isometric immersion.

4(c). Now, we identify the second fundamental form and the normal connection induced by f with II and ∇^E, respectively. This is done via Cartan's formalism for the isometric immersion f.

Let $f : (V \subset M, g) \to (\mathbb{R}^{n+k}, \tilde{g}_0)$ be the isometric immersion as above. We write $\tilde{\theta} = (\tilde{\theta}^1, \ldots, \tilde{\theta}^{n+k}) \in \Omega^1(\mathbb{R}^{n+k}) \cong C^\infty(\widetilde{M}; T^*\widetilde{M} \otimes T^*\widetilde{M})$ as the co-frame of $\{\widetilde{\partial}_a\}_1^{n+k}$. Recall from §2.3 that the GCR system for f are equivalent to the second structural system with respect to $\widetilde{\partial}$ or $\tilde{\theta}$. In particular, the corresponding connection 1-form on $\widetilde{M}$ for the Levi-Civita connection is

$$\widetilde{\mathcal{W}} = \begin{bmatrix} \tilde{\omega}^i_j & \tilde{\omega}^\alpha_i \\ \tilde{\omega}^i_\alpha & \tilde{\omega}^\beta_\alpha \end{bmatrix} = \begin{bmatrix} \tilde{\theta}^j(\tan\widetilde{\nabla}_\bullet \widetilde{\partial}_i) & \widetilde{S}_{\widetilde{\partial}_\alpha}\widetilde{\partial}_i \\ -\widetilde{S}_{\widetilde{\partial}_\alpha}\widetilde{\partial}_i & \tilde{\theta}^\beta(\nabla^\perp_\bullet \widetilde{\partial}_\alpha) \end{bmatrix}; \tag{A.12}$$

see Eq. (2.14). It satisfies

$$\begin{aligned}\widetilde{\mathcal{W}} = \{\tilde{\omega}^a_b\} \in & \Omega^1(\mathfrak{o}(\nu+\tau,(n+k)-(\nu+\tau))) \\ &= C^\infty(\widetilde{M}; T^*\widetilde{M} \otimes \mathfrak{o}(\nu+\tau,(n+k)-(\nu+\tau))),\end{aligned}$$

where $\widetilde{S}$ is the shape operator associated to f, and $\nabla^\perp$ is the projection of $\widetilde{\nabla}$ onto the normal bundle $TM^\perp$. Also, by the torsion-free condition of $\widetilde{\nabla}$, the first structural system (2.15) holds:

$$\mathrm{d}\tilde{\theta} = \tilde{\theta} \wedge \widetilde{\mathcal{W}}. \tag{A.13}$$

Therefore, by comparing the coordinate-wise representations of $\mathcal{W}$ and $\widetilde{\mathcal{W}}$, *i.e.*, Eqs. (2.14) and (A.12), in order to identify $(\widetilde{S}, \nabla^\perp)$ with (S, ∇^E), it suffices to establish

$$\mathcal{I}^*\widetilde{\mathcal{W}} = \mathcal{W}. \tag{A.14}$$

Indeed, we pullback Eq. (A.13) under $\mathcal{I}$. On one hand,

$$\mathcal{I}^*(\mathrm{d}\tilde{\theta}) = \mathrm{d}(\mathcal{I}^*\tilde{\theta}) = \mathrm{d}(f^*\tilde{\theta}) = \mathrm{d}\underset{\sim}{\Theta} = \underset{\sim}{\Theta} \wedge \mathcal{W}, \tag{A.15}$$

where we have used the commutativity of pullback and exterior differential, so that $\mathcal{I}$ respects the orthogonal splitting of $T\widetilde{M}$ and $TM \oplus E$, the duality of $\mathrm{d}f(\partial_i) = \widetilde{\partial}_i$, as well as the first structural system on $TM \oplus E$. On the other hand,

$$\mathcal{I}^*(\tilde{\theta} \wedge \widetilde{\mathcal{W}}) = \mathcal{I}^*\tilde{\theta} \wedge \mathcal{I}^*\widetilde{\mathcal{W}} = \underset{\sim}{\Theta} \wedge \mathcal{I}^*\widetilde{\mathcal{W}}, \tag{A.16}$$

owing to the distributivity of the pullback operation with respect to the wedge product, so that $\mathcal{I}^*\tilde{\theta} = f^*\tilde{\theta} = \underset{\sim}{\Theta}$ as above. Eq. (A.14) follows directly from Eqs. (A.15)–(A.16).

4(d). Finally, we prove the uniqueness of local isometric immersions up to rigid motions of the semi-Euclidean space. It is a direct consequence of the arguments in Step 3. Indeed, if $f' : (V, g) \to (\widetilde{M}, \tilde{g}_0)$ is another isometric immersion on V (a trivialized local chart) with $f'(q')$ given, then, for any local frame $\{\partial'_a\}$, we can take a rigid motion that transforms both q to q' and $\{\widetilde{\partial}_a\}$ to $\{\widetilde{\partial}'_a\}$; that is, a translation composed with an element of $O(\nu+\tau,(n+k)-(\nu+\tau))$. Then the argument follows from the uniqueness of solutions of the Pfaff system (which is based in turn on the uniqueness of the maximal integral submanifold found by the Frobenius theorem), as well as the uniqueness of solutions of the Poincaré system up to an additive constant.

We can now conclude the realization theorem in the C^∞ case from Steps 4(a)–4(d).

Appendix B Proof of Theorem 3.3

In this appendix, we prove Theorem 3.3 (the generalized quadratic theorem on LCA groups) for the theory of compensated compactness, a further extension of the classical quadratic theorem in [49,59]. First, we point out that the underlying strategy for the general case is similar to that in [49,59], in which separate estimates are derived in the Fourier space $\hat{G}$ for the *low-frequency region* (*i.e.*, in a compact set Ξ around 0) and the *high-frequency region* (*i.e.*, in the non-compact set $\hat{G} \setminus \Xi$). Assumptions (i)–(iii) are required only for controlling the high-frequency region. Notice that the high-frequency region always exists unless $\hat{G}$ is compact, which is equivalent to the condition that G is discrete, for which Theorem 3.3 trivially holds.

Proof of Theorem 3.3. By substituting u_ε with $u_\varepsilon - u$ as in the proof of Theorem 3.1, it suffices to assume that $u \equiv 0$. We divide the proof into five steps.

1. Since $u_\varepsilon \in L^2_c(G;\mathbb{C}^J)$ implies $u_\varepsilon \in L^1(G;\mathbb{C}^J)$, by the *Riemann–Lebesgue lemma* on LCA groups, we can find a compact set $\Xi \Subset \hat{G}$ such that $|\hat{u}_\varepsilon(\xi)| \leq \alpha$ for $\xi \in \hat{G}\backslash\Xi$, for each $\alpha > 0$ (*cf.* Tao [60]). In particular, $\sup\{|\hat{u}_\varepsilon(\xi)| \,:\, \xi \in \hat{G}\} \leq M$. On the other hand, for any $\phi \in L^2(G;\mathbb{C}^J)$, by the Plancherel formula, we have

$$\Big|\int_G u_\varepsilon \phi \,\mathrm{d}\mu_G\Big| = \Big|\int_{\hat{G}} \hat{u}_\varepsilon \hat{\phi} \,\mathrm{d}\mu_{\hat{G}}\Big|,$$

which converges to zero as $\varepsilon \to 0$ by assumption (C1). Thus, choosing $\hat{\phi} = \overline{\mathrm{sgn}(\hat{u}_\varepsilon)}\,\chi_\Xi$ with $\mathrm{sgn}(z) := \frac{z}{|z|}$ for $z \neq 0$, we obtain

$$\int_\Xi |\hat{u}_\varepsilon|^2 \,\mathrm{d}\mu_{\hat{G}} \leq M \int_\Xi |\hat{u}_\varepsilon| \,\mathrm{d}\mu_{\hat{G}} = M\Big|\int_{\hat{G}} \hat{u}_\varepsilon \hat{\phi} \,\mathrm{d}\mu_{\hat{G}}\Big| \longrightarrow 0.$$

Therefore, for the quadratic polynomial Q, we deduce

$$\int_\Xi |Q \circ \hat{u}_\varepsilon| \,\mathrm{d}\mu_{\hat{G}} \longrightarrow 0. \tag{B.1}$$

For the subsequent development, notice that there is a freedom of enlarging Ξ: It can be chosen as any large enough (with respect to $\mu_{\hat{G}}$) compact subset of $\hat{G}$ containing 0.

2. In this step, we establish the following claim:

Claim: *Given any $\delta > 0$ and any compact subset $\mathcal{K} \Subset \hat{G}$ such that $0 \notin \mathcal{K}$, there exists a constant $C_{\delta,\mathcal{K}} \in (0,\infty)$ so that, for any $\lambda \in \mathbb{C}^J$ and $\eta \in \mathcal{K}$,*

$$\mathrm{Re}\{Q(\lambda)\} \geq -\delta|\lambda|^2 - C_{\delta,\mathcal{K}}|m(\eta)(\lambda)|^2, \tag{B.2}$$

provided that $\mathrm{Re}(Q) \geq 0$ *on Λ_T. Meanwhile, under the same conditions for δ and $\mathcal{K}$,*

$$\mathrm{Im}\{Q(\lambda)\} \geq -\delta|\lambda|^2 - C_{\delta,\mathcal{K}}|m(\eta)(\lambda)|^2 \tag{B.3}$$

when $\mathrm{Im}(Q) \geq 0$ *on* Λ_T*. Notice that such a compact subset* $\mathcal{K}$ *exists, since* $\hat{G}$ *has locally compact topology.*

Indeed, observe that the claim holds for $\lambda = 0$. For $\lambda \neq 0$, we prove by contradiction. If the statement were false, there would exist $\delta_0 > 0$ such that, for each $n \in \mathbb{N}$, there exist $\lambda_n \in \mathbb{C}^J$ and $\eta_n \in \mathcal{K}$ so that

$$\mathrm{Re}\{Q(\lambda_n)\} < -\delta_0|\lambda_n|^2 - n|m(\eta_n)(\lambda_n)|^2. \tag{B.4}$$

Notice that this inequality is 2-homogeneous in λ_n; in particular, it is invariant under the scaling: $\lambda_n \mapsto c\lambda_n$ for any $c \in \mathbb{C} \setminus \{0\}$. Thus, without loss of generality, we may require $|\lambda_n| = 1$ for all n, so that $\{\lambda_n\}$ converges to some $\lambda_\infty \in \mathbb{C}^J$ of norm 1, after passing to a subsequence.

In this case, $|\mathrm{Re}(Q(\lambda_n))|$ is bounded uniformly in n (say, by C_0) so that

$$n|m(\eta_n)(\lambda_n)|^2 \leq C_0 - \delta_0.$$

This forces $|m(\eta_\infty)(\lambda_\infty)| = 0$, where $\eta_\infty \in \mathcal{K}$ is a limit of $\{\eta_n\}$, after passing to a further subsequence if necessary. Indeed, the subsequential convergence is guaranteed by the fact that $\hat{G}$ is Hausdorff, which is a part of the definition of LCA groups. The assumptions on $\mathcal{K}$ ensure that $\eta_\infty \neq 0$. Thus, by the definition of the cone in Eq. (3.19), $\lambda_\infty \in \Lambda_{\mathcal{T}}$. However, this implies

$$\mathrm{Re}\{Q(\lambda_\infty)\} \leq -\delta_0,$$

which contradicts the assumption that $\mathrm{Re}(Q) \geq 0$ on $\Lambda_{\mathcal{T}}$. Thus, the *claim* is proved for $\mathrm{Re}(Q)$. The arguments for $\mathrm{Im}(Q)$ are exactly the same, hence are omitted here.

3. Now, employing the *claim* in Step 3, we prove the following statement: *Whenever* $\mathrm{Re}(Q) \geq 0$ *on* $\Lambda_{\mathcal{T}}$*,*

$$\liminf_{\varepsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Re}(Q \circ \hat{u}_\varepsilon)\, \mathrm{d}\mu_{\hat{G}} \geq 0. \tag{B.5}$$

Similarly, for $\mathrm{Im}(Q) \geq 0$ *on* $\Lambda_{\mathcal{T}}$*,*

$$\liminf_{\varepsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Im}(Q \circ \hat{u}_\varepsilon)\, \mathrm{d}\mu_{\hat{G}} \geq 0. \tag{B.6}$$

To prove this statement, we invoke assumption (C2) on the Fourier multiplier. As $\{[\Phi^* m]\hat{u}_\varepsilon\}$ is pre-compact in $L^2(\hat{G};\mathbb{C}^J)$ and $\{\hat{u}_\varepsilon\}$ converges to zero weakly in L^2 (by the Plancherel formula), we have

$$\int_{\hat{G}\setminus\Xi} \left|m(\Phi(\xi))\hat{u}_\varepsilon(\xi)\right|^2 \mathrm{d}\mu_{\hat{G}}(\xi) \longrightarrow 0.$$

Take $\eta = \Phi(\xi) \in \mathcal{K}$ and $\lambda = \hat{u}_\varepsilon(\xi) \in \mathbb{C}^J$ in Eq. (B.2) in Step 2. It shows that, for each $\delta > 0$, there exists $0 < C_{\delta,\mathcal{K}} < \infty$ such that

$$\mathrm{Re}(Q \circ \hat{u}_\varepsilon(\xi)) > -\delta|\hat{u}_\varepsilon(\xi)|^2 - C_{\delta,\mathcal{K}}\left|m(\Phi(\xi))\hat{u}_\varepsilon(\xi)\right|^2 \qquad \text{for } \xi \in \hat{G} \setminus \Xi.$$

Then, integrating over $\hat{G} \setminus \Xi$ and sending $\varepsilon \to 0$, we have

$$\liminf_{\varepsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Re}(Q \circ \hat{u}_\varepsilon)\, \mathrm{d}\mu_{\hat{G}} \geq -\delta \sup_{\varepsilon\geq 0} \|\hat{u}_\varepsilon\|^2_{L^2(\hat{G}\setminus\Xi)} \geq -\delta M$$

for a universal constant $M < \infty$, where we have used the precompactness of $\{\hat{u}_\varepsilon\}$ in $L^2(\hat{G}; \mathbb{C}^J)$, which is implied by assumption (C1) and the Plancherel formula. As $\delta > 0$ is arbitrary, Eq. (B.5) is proved. The proof for the imaginary part, *i.e.*, Eq. (B.6), holds analogously.

4. To conclude the theorem, note that, by changing $Q \mapsto -Q$ in Eq. (B.5), the following inequality holds:

$$\limsup_{\varepsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Re}(Q \circ \hat{u}_\varepsilon)\, \mathrm{d}\mu_{\hat{G}} \leq 0 \qquad \text{for } \mathrm{Re}(Q) \leq 0 \text{ on } \Lambda_{\mathcal{T}}. \tag{B.7}$$

By assumption (C3), *i.e.*, $\mathrm{Re}(Q) = 0$ on $\Lambda_{\mathcal{T}}$, inequality (B.7) together with (B.5) verifies the assertion outside a compact set Ξ, *i.e.*, $\lim_{\varepsilon\to 0} \int_{\hat{G}\setminus\Xi} \mathrm{Re}(Q \circ \hat{u}_\varepsilon)\, \mathrm{d}\mu_{\hat{G}} = 0$. Moreover, in Step 1, the same result on Ξ has been established in Eq. (B.1). Thus, in view of the Plancherel formula, we have

$$\lim_{\varepsilon\to 0} \int_G \mathrm{Re}(Q \circ u_\varepsilon)\, \mathrm{d}\mu_G = 0.$$

As in Steps 2–3, the analogous statement for $\mathrm{Im}(Q \circ u_\varepsilon)$ can be established similarly. This completes the proof.

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Conflict of interest

The authors declare that they have no conflict of interest.

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